A Landau fluid model for dispersive magnetohydrodynamics

T. Passot and P. L. Sulem
Observatoire de la Côte d'Azur, CNRS, Boîte Postale 4229, 06304 Nice, Cedex 4, France

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A monofluid model with Landau damping is presented for strongly magnetized electron-proton collisionless plasmas whose distribution functions are close to bi-Maxwellians. This description that includes dynamical equations for the gyrotropic components of the pressure and heat flux tensors, extends the Landau-fluid model of Snyder, Hammett, and Dorland [Phys. Plasmas **4**, 3974 (1997)] by retaining Hall effect and finite Larmor radius corrections. It accurately reproduces the weakly nonlinear dynamics of dispersive Alfvén waves whose wavelengths are large compared to the ion inertial length, whatever their direction of propagation, and also the rapid Landau dissipation of long magnetosonic waves in a warm plasma. *© 2004 American Institute of Physics*.
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I. INTRODUCTION

Both in natural and fusion plasmas, collisions are generally negligible, making the usual magnetohydrodynamics questionable. On the other hand, in most situations, direct numerical integrations of the Vlasov–Maxwell equations in three space dimensions are beyond the capabilities of the present day computers when a broad range of scales are involved. The gyrokinetic description[1,2] that averages over the gyrotropic motion of the particles and that is extensively used for fusion plasmas, reduces the number of independent variables but still needs an enormous computational strength.

Situations involving a broad range of scales require a formalism that preserves most of the aspects of a fluid description but includes realistic approximations of the pressure and heat flux tensors. The effect of wave-particle resonances that provide the dominant dissipation processes should in particular be retained. In a collisionless plasma, a fluid behavior can only result from collective constraints, such as the presence of a strong magnetic field. In this case, Chew, Goldberger, and Low (CGL) (Ref. 3) first proposed the "double adiabatic laws" or CGL equations for the parallel and perpendicular gyrotropic pressure components, where all the heat fluxes are neglected. The conditions of validity of this assumption are rather stringent.[4] The onset of the mirror instability is, for example, not correctly described[5] within this approximation that requires a phase velocity much larger than the thermal velocity of the particles. Closures that reproduce linear results from kinetic theory were also proposed but they depend on the equilibrium state and are often presented in Fourier space, leading to the definition of effective polytropic indices.[6,7] In the context of fusion plasmas, an extensive literature was devoted during the last decades to the gyrofluid description[8,9] based on the evolution of hydrodynamic moments obtained from the gyrokinetic equations, and thus also written in a local coordinate system. A hybrid description of low frequency phenomena involving the coupling of a monofluid description with pressure tensors for ions and electrons prescribed by gyrokinetic equations was also developed.[10]

A simplified description more easily amenable to large-scale numerical simulations of a collisionless plasma permeated by a strong magnetic field was suggested by Hammett and co-workers in the form of Landau fluids built to account for wave-particle resonance effects within a magnetohydro dynamics (MHD) framework. The full electromagnetic case is presented by Snyder, Hammett, and Dorland,[11] hereafter referred to as SHD. Hydrodynamic equations for the density and velocity of the plasma are obtained by taking moments of the microscopic equations. SHD start from guiding center equations but an equivalent derivation can be made from the Vlasov–Maxwell system. The resulting hierarchy must nevertheless be closed and the main work consists in a proper determination of the pressure tensor associated with each species. For the sake of simplicity, an electron-proton plasma is considered in a simple geometry (no curvature drift), with an homogeneous equilibrium state characterized by bi-Maxwellian distribution functions. In its original presentation, the model is limited to scales large enough for both Hall effect and finite Larmor radius (FLR) corrections to be totally negligible. As shown by SHD, this description predicts the correct threshold of the mirror instability. A generalization is, however, needed in order to consider dispersive MHD turbulence.

Our goal is thus to develop a simple monofluid model able to accurately reproduce the weakly nonlinear dynamics of most MHD waves, including kinetic Alfvén waves with transverse wave number small compared with the inverse proton inertial length. These waves, characterized by an angle of propagation α such that $\cos^2\alpha \ll \beta$ (where β denotes the squared ratio of the ion acoustic to the Alfvén speeds), are supposed to be produced by the quasi-two-dimensional energy cascade that develops in Alfvén wave turbulence. A simplified model was recently derived[12] and benchmarked by direct comparisons with Vlasov–Maxwell predictions in the limit of long-wavelength small-amplitude perturbations. For parallel Alfvén waves, a reductive perturbative expansion of this model reproduces the kinetic derivative nonlinear Schrödinger (KDNLS) equation[13–15] (including its extension to multidimensional wave trains[16]) derived from the Vlasov–Maxwell equations, up to the replacement of the plasma response function by its two- or four-pole Padé approximants. For magnetosonic waves,[17] agreement was obtained with the phase velocity and the Landau damping given in the literature[18] for the regime $(m_e/m_p) \ll \beta \ll (T_e/T_p)$ of adiabatic protons and isothermal electrons with isotropic temperatures. It, however, turns out that the description of oblique and kinetic Alfvén waves requires a more refined description of the finite Larmor radius effects associated with the nongyrotropic contributions of both the pressure and heat flux tensors. A first extension[17] of this model was presented in the regime of adiabatic protons and isothermal electrons with small β, that reproduces the classial dispersion and Landau damping of kinetic Alfvén waves in this regime.[18,19] This approach actually involves a heuristic closure relation for the electron pressure that is here recovered as a limiting case of a more general Landau-fluid model whose derivation is the main object of the present paper.

In Sec. II, the scalings associated with the various MHD waves are explicited and a monofluid description of the plasma is obtained, under conditions consistent with the weakly nonlinear regime. In order to describe anisotropic situations, the pressure tensor of each particle species is retained. It includes gyrotropic components that evolve on hydrodynamic time scales, together with nongyrotropic ones that rapidly adjust to the variations of the hydrodynamic quantities ("slaved" dynamics) and are amenable to a perturbative description (Sec. III). In Sec. IV, general closure approximations for the gytrotropic and nongyrotropic heat fluxes are inferred from the kinetic theory of long oblique Alfvén waves presented in Appendices A–C. As mentioned above, this regime that retains the kinetic effects to leading order, can indeed be viewed as a distinguished limit covering more general situations. The resulting model and its validation are presented in Sec. V. A few conclusions and projects for further developments are briefly presented in the last section.

II. AN ASYMPTOTIC FRAMEWORK FOR A FLUID DESCRIPTION

A. The small amplitude regime

The usual procedure[5] to describe the dynamics of a strongly magnetized collisionless plasma at scales large compared to those of the ion gyromotion consists in performing an asymptotics (referred to as a $1/\Omega_p$ expansion) where the small parameter is the ratio of the typical considered frequency to the ion gyrofrequency. This approach is appropriate when no smallness assumption is made on the amplitude of the fluctuations, but may be conflicting with the weak-nonlinearity ordering required to close the moment hierarchy. When addressing the weakly nonlinear regime, it is thus preferable to use a unique expansion parameter to characterize the small amplitudes and the long-wavelengths and low frequencies of the perturbations. In the distinguished limit that ensures the balance of the nonlinear and dispersive effects, a reductive perturbative expansion then leads to the classical long-wave equations (such as Korteweg–de Vries or derivative nonlinear Schrödinger). This asymptotics may retain terms which are subdominant in an $1/\Omega_p$ expansion that is relative to the scale separation only.

The fluid equations to be derived in this paper are requested to correctly capture the weakly nonlinear dynamics of dispersive MHD waves, by fitting with the kinetic theory within the ordering prescribed by a reductive perturbative expansion. This approach has the main advantage to separate the various types of waves, retaining only those terms that contribute to their dynamics. It also provides a rigorous framework for a nonlinear theory where some terms are evaluated at the linear level, as, for example, requested at the level of the heat flux closure.

B. The MHD wave scalings

In a reductive perturbative expansion, the various MHD waves are selected by prescribing different orderings. The ambient magnetic field being taken in the z direction, we assume a propagation in the (x,z) plane along an axis z' making an angle α with the ambient field. For perturbations depending only on z' and propagating at velocity V_0, we define the stretched coordinate $\xi = \epsilon^{1/2}(z' - V_0 t)$.

1. Oblique magnetosonic waves

The magnetosonic waves are selected by prescribing $b_x = \epsilon b_x^{(1)} + \cdots$, $b_y = \epsilon^{3/2} b_y^{(1)} + \cdots$, $b_z = B_0 + \epsilon b_z^{(1)} + \cdots$, $\rho = \rho^{(0)} + \epsilon \rho^{(1)} + \cdots$, $u_x = \epsilon u_x^{(1)} + \cdots$, $u_y = \epsilon^{3/2} u_y^{(1)} + \cdots$, $u_z = \epsilon u_z^{(1)} + \cdots$, $p_{\perp r} = p_{\perp r}^{(0)} + \epsilon p_{\perp r}^{(1)} + \cdots$, $p_{\parallel r} = p_{\parallel r}^{(0)} + \epsilon p_{\parallel r}^{(1)} + \cdots$, where as usual b is the magnetic field, ρ and u are the density and velocity of the plasma, $p_{\perp r}$ and $p_{\parallel r}$ are the transverse and parallel pressures of the particles of species r. The dispersion and the nonlinearities then act on a slow time $\tau = \epsilon^{3/2} t$.

2. Oblique Alfvén waves

The reductive perturbative expansion now involves the scalings $b_y = \epsilon^{1/2}(b_y^{(1)} + \epsilon b_y^{(2)} + \cdots)$, $u_y = \epsilon^{1/2}(u_y^{(1)} + \epsilon u_y^{(2)} + \cdots)$, while the previously defined scalings are retained for the other quantities.

3. Parallel Alfvén waves

In the case of a propagation angle $\alpha = 0$, one prescribes $b_x = \epsilon^{1/4} b_x^{(1)} + \cdots$, $b_y = \epsilon^{1/4} b_y^{(1)} + \cdots$, $b_z = B_0 + \epsilon^{1/2} b_z^{(1)} + \cdots$, $\rho = \rho^{(0)} + \epsilon^{1/2} \rho^{(1)} + \cdots$, $u_x = \epsilon^{1/4} u_x^{(1)} + \cdots$, $u_y = \epsilon^{1/4} u_y^{(1)} + \cdots$, $u_z = \epsilon^{1/2} u_z^{(1)} + \cdots$, $p_{\perp r} = p_{\perp r}^{(0)} + \epsilon^{1/2} p_{\perp r}^{(1)} + \cdots$, $p_{\parallel r} = p_{\parallel r}^{(0)} + \epsilon^{1/2} p_{\parallel r}^{(1)} + \cdots$, with a slow time $\tau = \epsilon t$.

Furthermore, for all the waves, the gyrotropic heat fluxes are scaled similarly to the pressures. The magnitude of the nongyrotropic components will be explicited later on, when these contributions will be considered (Sec. IV A).

The above scalings indicate that nonlinear effects comparable to dispersion occur with an amplitude that is smaller for magnetosonic waves than for Alfvén waves. This reflects the longitudinal character of the former waves for which a relatively strong dispersion is requested to arrest shock formation. In contrast, parallel Alfvén waves can support much larger amplitude since they are incompressible. It follows that a weakly nonlinear theory of magnetosonic waves requires a higher order perturbation theory, as it will be discussed in more details in Sec. V. This delicate situation can nevertheless be prevented by the presence of Landau damping that, when the β of the plasma is not too small, acts on the shortest time scale, making the nonlinear and dispersive corrections subdominant. The small β limit that makes the electron inertia relevant is in any case out of the scope of a monofluid theory. For larger wave amplitude, the lowest order corrections in the usual $1/\Omega_p$ expansion together with a simple Landau-fluid closure for the gyrotropic pressures[11,12] are sufficient. Furthermore, an expansion valid for oblique Alfvén waves, where both the Hall term and nongyrotropic heat flux components enter at dominant order, will retain all the relevant terms for parallel waves (with possible additional subdominant corrections) and also for magnetosonic waves in the most usual situations. As a consequence of these observations, the construction of the monofluid model will be based on the weakly nonlinear dynamics of oblique Alfvén waves with typical wavelengths large compared to the proton inertial length. This approach involves several steps.

C. From a bifluid to a monofluid description

Starting from the Vlasov–Maxwell equations (A1)–(A4) and writing the equations satisfied by the successive moments of the distribution function for particles of species r, one derives an exact hierarchy of fluid equations for the corresponding density $\rho_r=m_r n_r\int f_r d^3v$, hydrodynamic velocity $u_r=\int v f_r d^3v\big/\int f_r d^3v$, pressure tensor $\mathbf{P}_r=m_r n_r\int(v-u_r)\otimes(v-u_r)f_r d^3v$ and heat flux tensor $\mathbf{Q}_r=m_r n_r\int(v-u_r)\otimes(v-u_r)\otimes(v-u_r)f_r d^3v$, in the usual form[5]

$$\partial_t\rho_r+\nabla\cdot(\rho_r u_r)=0, \tag{1}$$

$$\partial_t u_r+u_r\cdot\nabla u_r+\frac{1}{\rho_r}\nabla\cdot\mathbf{P}_r-\frac{q_r}{m_r}\left(e+\frac{1}{c}u_r\times b\right)=0, \tag{2}$$

$$\partial_t\mathbf{P}_r+\nabla\cdot(u_r\mathbf{P}_r+\mathbf{Q}_r)+\left[\mathbf{P}_r\cdot\nabla u_r+\frac{q_r}{m_r c}b\times\mathbf{P}_r\right]^S=0, \tag{3}$$

where the tensor $b\times\mathbf{P}_r$ has elements $(b\times\mathbf{P}_r)_{ij}=\epsilon_{iml}b_m P_{r\ lj}$ and where, for a square matrix $\mathbf{A}$, one defines $\mathbf{A}^S=\mathbf{A}+\mathbf{A}^{\mathrm{tr}}$. One has $(b\times\mathbf{P}_r)^{\mathrm{tr}}=-\mathbf{P}_r\times b$. In order to distinguish between scalar and tensorial pressures, bold letters are used to denote tensors of rank two and higher. Coupled to Maxwell equations, such a multifluid description resolves the small spatiotemporal scales associated with Langmuir waves that are unneeded when concentrating on the large-scale dynamics of dispersive MHD waves. A monofluid description together with the additional approximation of neglecting electron inertia, allows the filtering of these small scales. One is thus led to consider the plasma velocity $u=(1/\rho)\Sigma_r\rho_r u_r$, where $\rho=\Sigma_r\rho_r$ is the plasma density, and to define the pressure and heat flux tensors associated with each particle species in terms of the deviations from this barycentric velocity, in the form $\mathbf{p}_r=m_r n_r\int(v-u)\otimes(v-u)f_r d^3v$ and $\mathbf{q}_r=m_r n_r\int(v-u)\otimes(v-u)\otimes(v-u)f_r d^3v$. One has

$$\mathbf{P}_r=\mathbf{p}_r-\rho_r(u-u_r)\otimes(u-u_r) \tag{4}$$

and

$$Q_{r\ ijk}=q_{r\ ijk}+p_{r\ ij}(u-u_r)_k+p_{rik}(u-u_r)_j+p_{rjk}(u-u_r)_i, \tag{5}$$

where the subscripts ijk refer to components of the corresponding tensors.

Defining $\boldsymbol{\delta}_{\mathbf{r}}=\mathbf{P}_r-\mathbf{p}_r$ and

$$\mathbf{R}_r=\nabla\cdot(u_r\boldsymbol{\delta}_{\mathbf{r}})+[\boldsymbol{\delta}_r\cdot\nabla u_r+(\nabla\cdot\mathbf{P}_r)\otimes(u-u_r)]^S, \tag{6}$$

one has in Eq. (3)

$$\begin{aligned}\nabla\cdot(u_r\mathbf{P}_r+\mathbf{Q}_r)+[\mathbf{P}_r\cdot\nabla u_r]^S\\=\nabla\cdot(u\mathbf{p}_r+\mathbf{q}_r)+[\mathbf{p}_r\cdot\nabla u]^S+\mathbf{R}_r.\end{aligned} \tag{7}$$

For the orderings involved in the reductive perturbative analysis of the various MHD waves discussed above, neglecting $\boldsymbol{\delta}_r$ and $\mathbf{R}_r$ contributions in the equation for $\mathbf{p}_r$ is possible if the expansion of this quantity is limited to orders strictly lower than ϵ^2 for oblique Alfvén waves and $\epsilon^{5/2}$ for magnetosonic waves. This leads us to replace Eq. (3) by

$$\partial_t\mathbf{p}_r+\nabla\cdot(u\mathbf{p}_r+\mathbf{q}_r)+\left[\mathbf{p}_r\cdot\nabla u+\frac{q_r}{m_r c}b\times\mathbf{p}_r\right]^S=0. \tag{8}$$

Furthermore, one easily gets that

$$\partial_t\rho+\nabla\cdot(u\rho)=0 \tag{9}$$

and

$$\partial_t(\rho u)+\nabla\cdot(\rho u\otimes u)+\nabla\cdot\mathbf{p}-\frac{1}{c}j\times b=0, \tag{10}$$

where $\mathbf{p}=\Sigma_r\mathbf{p}_r$ denotes the total pressure tensor and where the electric current $j=\Sigma_r q_r n_r\int v f_r d^3v=\Sigma_r(q_r/m_r)\rho_r u_r$ is given by $j=c/4\pi\nabla\times b$. In this derivation, we neglect the displacement current and also make the approximation of quasineutrality $\Sigma_r q_r\rho_r/m_r=0$, as usual when considering slow motion of fluid elements of size greater than the Debye length.[20]

The current j obeys

$$\begin{aligned}\partial_t j+\nabla\cdot\left(\sum_r\frac{q_r\rho_r}{m_r}u_r\otimes u_r\right)+\sum_r\frac{q_r}{m_r}\nabla\cdot\mathbf{P}_r\\-\sum_r\frac{q_r^2\rho_r}{m_r^2}\left(e+\frac{1}{c}u_r\times b\right)=0.\end{aligned} \tag{11}$$

Using the identity

$$\begin{aligned}\sum_r\frac{q_r\rho_r}{m_r}u_r\otimes u_r=u\otimes j+j\otimes u-\sum_r\frac{q_r\rho_r}{m_r}u\otimes u\\+\sum_r\frac{q_r\rho_r}{m_r}(u_r-u)\otimes(u_r-u),\end{aligned} \tag{12}$$

and the fact that for a plasma of protons and electrons of electric charge $q_p=-q_e=q$,

$$\sum_r \frac{q_r^2}{m_r^2}\rho_r u_r = q^2\left(\frac{1}{m_e}+\frac{1}{m_p}\right)\frac{\rho u}{m_e+m_p} - q\left(\frac{m_p}{m_e}-\frac{m_e}{m_p}\right)\frac{j}{m_e+m_p}, \quad (13)$$

one gets[20]

$$\partial_t j + \nabla \cdot \left(u\otimes j + j\otimes u - \sum_r \frac{q_r\rho_r}{m_r}u\otimes u\right) + \sum_r \frac{q_r}{m_r}\nabla \cdot \mathbf{p}_r - \sum_r \frac{q_r^2\rho_r}{m_r^2}e - \frac{q^2}{c}\left(\frac{1}{m_e}+\frac{1}{m_p}\right)\frac{\rho u\times b}{m_e+m_p} + \frac{q}{c}\left(\frac{m_p}{m_e}-\frac{m_e}{m_p}\right)\frac{j\times b}{m_e+m_i} = 0. \quad (14)$$

This equation simplifies when terms involving the ratio m_e/m_p are neglected and quasineutrality is assumed, which leads to write $\rho_r = m_r n$ and $u \approx u_p$. One obtains

$$\partial_t j + \nabla \cdot (u\otimes j + j\otimes u) - \frac{q^2 n}{m_e}\left(e + \frac{u\times b}{c} - \frac{j\times b}{nqc} + \frac{1}{qn}\nabla \cdot \mathbf{p}_e\right) = 0. \quad (15)$$

For small nonlinearity and under the assumption $\beta \gg (m_e/m_i)$, the two first terms of the above equation are subdominant. From Maxwell equation (A2), one then obtains the induction equation

$$\partial_t b - \nabla \times (u\times b) = -\frac{cm_p}{q}\nabla \times \left[\frac{1}{4\pi\rho}(\nabla\times b)\times b - \frac{1}{\rho}\nabla \cdot \mathbf{p}_e\right], \quad (16)$$

that includes the Hall term together with the effect of the electron pressure.

Equations (9), (10), (8), and (16) constitute a closed system, provided a closure approximation is made to express the heat fluxes. Nevertheless, a direct resolution of Eq. (8) would have to resolve time scales associated with the gyromotion of the particles, a condition that is practically impossible to achieve in numerical simulations that also retain hydrodynamical scales. As shown in Sec. III, this scale separation can in fact be used to define a reduced description where the evolution of the gyrotropic components of the pressure tensors is followed on hydrodynamic time scales, while the nongyrotropic ones obey a slaved dynamics in the sense that they are prescribed by the instantaneous values of hydrodynamic quantities. A similar separation can be made at the level of the heat fluxes that contribute to the gyrotropic pressures through both gyrotropic and nongyrotropic components. Again the gyrotropic heat fluxes require a closure approximation taking the form of dynamical equations, while the nongyrotropic ones are slaved (Sec. IV).

III. THE PRESSURE TENSORS

In order to isolate the gyrotropic components of the pressure tensor, it is convenient to rewrite Eq. (8) for the pressure tensor of each particle species in the form

$$\mathbf{p}_r \times \hat{b} - \hat{b}\times \mathbf{p}_r = \mathbf{k}_r, \quad (17)$$

where $\hat{b}=b/|b|$ is the unit vector along the local magnetic field and

$$\mathbf{k}_r = \frac{1}{\Omega_r}\frac{B_0}{|b|}\left[\frac{d\mathbf{p}_r}{dt} + (\nabla\cdot u)\mathbf{p}_r + \nabla \cdot \mathbf{q}_r + (\mathbf{p}_r\cdot\nabla u)^S\right]. \quad (18)$$

In this equation, B_0 denotes the amplitude of the ambient field and $\Omega_r = (q_r B_0/m_r c)$ is the gyrofrequency of the particles of species r. Furthermore, $d/dt = \partial_t + u\cdot\nabla$ denotes the convective derivative.

A few classical results are first recalled for completeness.[21,22] We first note that the left-hand side of Eq. (17) can be viewed as a self-adjoint linear operator acting on $\mathbf{p}_r$, whose kernel is spanned by the tensors $(\mathbf{I}-\hat{b}\otimes\hat{b})$ and $\hat{b}\otimes\hat{b}$. It is thus convenient to define the projection $\overline{\mathbf{a}}$ of any (3×3) rank two tensor $\mathbf{a}$ on the image of this operator as

$$\overline{\mathbf{a}} = \mathbf{a} - \frac{1}{2}\mathbf{a}:(\mathbf{I}-\hat{b}\otimes\hat{b})(\mathbf{I}-\hat{b}\otimes\hat{b}) - (\mathbf{a}:\hat{b}\otimes\hat{b})\hat{b}\otimes\hat{b}, \quad (19)$$

which implies $\mathrm{tr}\,\overline{\mathbf{a}}=0$ and $\overline{\mathbf{a}}:\hat{b}\otimes\hat{b}=0$. Here $\mathbf{I}$ is the identity matrix and the double contraction of two square matrices $\mathbf{m}$ and $\mathbf{n}$ is defined as $\mathbf{m}:\mathbf{n}=\Sigma_{ij}m_{ij}n_{ij}$. In particular, the pressure tensor is written as the sum $\mathbf{p}_r = \mathbf{p}_r^G + \boldsymbol{\pi}_r$ of an element of the kernel

$$\mathbf{p}_r^G = \frac{1}{2}\mathbf{p}_r:(\mathbf{I}-\hat{b}\otimes\hat{b})(\mathbf{I}-\hat{b}\otimes\hat{b}) + (\mathbf{p}_r:\hat{b}\otimes\hat{b})\hat{b}\otimes\hat{b} \quad (20)$$

$$\equiv p_{\perp r}(\mathbf{I}-\hat{b}\otimes\hat{b}) + p_{\parallel r}\hat{b}\otimes\hat{b} \quad (21)$$

and of a nongyrotropic component $\boldsymbol{\pi}_r = \overline{\mathbf{p}_r}$ that thus satisfies $\mathrm{tr}\,\boldsymbol{\pi}_r = 0$ and $\boldsymbol{\pi}_r:\hat{b}\otimes\hat{b}=0$.

A. Dynamics of the gyrotropic pressures

To obtain the equations for the gyrotropic pressure components, one applies the trace and the contraction with $\hat{b}\otimes\hat{b}$ on both sides of Eq. (17) to get

$$\mathrm{tr}\frac{d\mathbf{p}_r}{dt} + (\nabla\cdot u)\mathrm{tr}\,\mathbf{p}_r^G + \mathrm{tr}\,(\nabla\cdot\mathbf{q}_r) + \mathrm{tr}\,(\mathbf{p}_r^G\cdot\nabla u)^S + s_{1r} = 0 \quad (22)$$

with $s_{1r} = \mathrm{tr}\,(\boldsymbol{\pi}_r\cdot\nabla u)^S$ and

$$\frac{d\mathbf{p}_r}{dt}:\hat{b}\otimes\hat{b} + [(\nabla\cdot u)\mathbf{p}_r^G + \nabla\cdot\mathbf{q}_r + (\mathbf{p}_r^G\cdot\nabla u)^S]:\hat{b}\otimes\hat{b} + s_{2r} = 0 \quad (23)$$

with $s_{2r} = (\boldsymbol{\pi}_r\cdot\nabla u)^S:\hat{b}\otimes\hat{b}$.

The trace and the time derivative commute but this is not the case for the contraction with $\hat{b}\otimes\hat{b}$. One writes

$$\frac{d\mathbf{p}_r}{dt}:\hat{b}\otimes\hat{b}=\frac{d}{dt}(\mathbf{p}_r:\hat{b}\otimes\hat{b})-\mathbf{p}_r:\frac{d}{dt}(\hat{b}\otimes\hat{b})=\frac{dp_{\parallel r}}{dt}-s_{3r}, \tag{24}$$

where, using $(d\hat{b}/dt\otimes\hat{b}+\hat{b}\otimes d\hat{b}/dt):(\mathbf{I}-\hat{b}\otimes\hat{b})=0$ and $(d\hat{b}/dt\otimes\hat{b}+\hat{b}\otimes d\hat{b}/dt):\hat{b}\otimes\hat{b}=0$, one has $s_{3r}=\boldsymbol{\pi}_r:d/dt(\hat{b}\otimes\hat{b})$. One thus gets generalized CGL equations that include heat fluxes and coupling to the nongyrotropic components of the pressure tensors,

$$\partial_t p_{\perp r}+\nabla\cdot(u\,p_{\perp r})+p_{\perp r}\nabla\cdot u-p_{\perp r}\hat{b}\cdot\nabla u\cdot\hat{b}+\tfrac{1}{2}[\mathrm{tr}\,\nabla\cdot\mathbf{q}_r-\hat{b}\cdot(\nabla\cdot\mathbf{q}_r)\cdot\hat{b}]+\tfrac{1}{2}(s_{1r}-s_{2r}+s_{3r})=0, \tag{25}$$

$$\partial_t p_{\parallel r}+\nabla\cdot(u\,p_{\parallel r})+2p_{\parallel r}\hat{b}\cdot\nabla u\cdot\hat{b}+\hat{b}\cdot(\nabla\cdot\mathbf{q}_r)\cdot\hat{b}+s_{2r}-s_{3r}=0. \tag{26}$$

One easily checks that for the scalings defined in Sec. II A and the nongyrotropic pressure components given in Sec. III B, the couplings s_{1r}, s_{2r}, and s_{3r} to the nongyrotropic pressure components are negligible. Note that similar equations for the gyrotropic pressures can be obtained in a bifluid description, up to the replacement of the plasma velocity u by that of the individual species u_r. It is noticeable that in the present derivation based on the hypothesis of weak nonlinearity together with long spatial and temporal scales, the parallel and transverse pressures decouple from the nongyrotropic pressure components but are sensitive to the gyrotropic and nongyrotropic components of the heat fluxes $\mathbf{q}_r$ that can both contribute to the gyrotropic components of $\nabla\cdot\mathbf{q}_r$.

B. Nongyrotropic pressure contributions

In order to determine the nongyrotropic contributions to the pressure tensor of the various particle species, we start from Eq. (17). Using the solvability conditions provided by the equations for the gyrotropic pressures, it is rewritten

$$\boldsymbol{\pi}_r\times\hat{b}-\hat{b}\times\boldsymbol{\pi}_r=\overline{\mathbf{k}_r}, \tag{27}$$

where $\mathbf{k}_r$ can be decomposed into the sum of a contribution involving the gyrotropic pressures and the heat fluxes

$$\boldsymbol{\kappa}_r=\frac{1}{\Omega_r}\frac{B_0}{|b|}\left[\frac{d\mathbf{p}_r^G}{dt}+(\nabla\cdot u)\mathbf{p}_r^G+\nabla\cdot\mathbf{q}_r+(\mathbf{p}_r^G\cdot\nabla u)^S\right] \tag{28}$$

and of a term linear in $\boldsymbol{\pi}_r$

$$L(\boldsymbol{\pi}_r)=\frac{1}{\Omega_r}\frac{B_0}{|b|}\left[\frac{d\boldsymbol{\pi}_r}{dt}+(\nabla\cdot u)\boldsymbol{\pi}_r+(\boldsymbol{\pi}_r\cdot\nabla u)^S\right]. \tag{29}$$

In $\overline{\boldsymbol{\kappa}_r}$, the second term of the right-hand side (RHS) of Eq. (28) does not contribute, while the first one rewrites

$$\overline{\frac{d\mathbf{p}_r^G}{dt}}=(p_{\parallel r}-p_{\perp r})\frac{d}{dt}(\hat{b}\otimes\hat{b})=(p_{\parallel r}-p_{\perp r})\frac{1}{|b|^2}\left(\frac{db}{dt}\otimes b+b\otimes\frac{db}{dt}-\frac{2}{|b|}\frac{d|b|}{dt}b\otimes b\right) \tag{30}$$

that is explicited by using the induction equation (16).

It is then convenient to split the nongyrotropic pressure as $\boldsymbol{\pi}_r=\boldsymbol{\pi}_{r,1}+\boldsymbol{\pi}_{r,2}$ with

$$\boldsymbol{\pi}_{r,1}\times\hat{b}-\hat{b}\times\boldsymbol{\pi}_{r,1}=\overline{\boldsymbol{\kappa}_r}, \tag{31}$$

$$\boldsymbol{\pi}_{r,2}\times\hat{b}-\hat{b}\times\boldsymbol{\pi}_{r,2}=\overline{L(\boldsymbol{\pi}_{r,1})}+\overline{L(\boldsymbol{\pi}_{r,2})}. \tag{32}$$

In a weakly nonlinear regime, the quantity $L(\boldsymbol{\pi}_r)$ is of higher order than $\boldsymbol{\pi}_r$, which enables one to neglect $L(\boldsymbol{\pi}_{r,2})$ in Eq. (32). We restrict ourselves to this level of approximation since pushing further the above perturbative calculation would conflict with the approximations made for the derivation of the pressure equation (8) used in a monofluid description. The above equations can be solved in the form[21]

$$\boldsymbol{\pi}_{r,1}=\tfrac{1}{4}[\hat{b}\times\overline{\boldsymbol{\kappa}_r}\cdot(\mathbf{I}+3\hat{b}\otimes\hat{b})]^S, \tag{33}$$

$$\boldsymbol{\pi}_{r,2}=\tfrac{1}{4}[\hat{b}\times\overline{L(\boldsymbol{\pi}_{r,1})}\cdot(\mathbf{I}+3\hat{b}\otimes\hat{b})]^S, \tag{34}$$

where the overlines turn out not to be necessary in the above formulae. These expressions are nevertheless cumbersome to be used in a numerical code.

In some situations, the contribution $\boldsymbol{\pi}_{r,1}$ is sufficient and can even be simplified by approximating $\hat{b}$ by the unit vector $\hat{z}$ along the ambient magnetic field. This leads to define $\boldsymbol{\pi}_r^{[1]}$ by

$$\boldsymbol{\pi}_r^{[1]}\times\hat{z}-\hat{z}\times\boldsymbol{\pi}_r^{[1]}=\overline{\overline{\boldsymbol{\chi}_r^{[1]}}} \tag{35}$$

together with $\hat{z}\cdot\boldsymbol{\pi}_r^{[1]}\cdot\hat{z}=0$ and $\boldsymbol{\pi}_r^{[1]}:\mathbf{I}=0$, where

$$\boldsymbol{\chi}_r^{[1]}=\frac{1}{\Omega_r}\left[\left(\frac{d\mathbf{p}_r^G}{dt}\right)^{[1]}+(\mathbf{p}_r^{G[1]}\cdot\nabla u)^S+\nabla\cdot\mathbf{q}_r\right] \tag{36}$$

with $\mathbf{p}_r^{G[1]}=p_{\perp r}(\mathbf{I}-\hat{z}\otimes\hat{z})+p_{\parallel r}\hat{z}\otimes\hat{z}$ and

$$\left(\frac{d\mathbf{p}_r^G}{dt}\right)^{[1]}=\frac{dp_{\perp r}}{dt}(\mathbf{I}-\hat{z}\otimes\hat{z})+\frac{dp_{\parallel r}}{dt}\hat{z}\otimes\hat{z}+(p_{\parallel p}-p_{\perp p})\partial_z[u\otimes\hat{z}-(\hat{z}\cdot u)\hat{z}\otimes\hat{z}]^S. \tag{37}$$

We here denote by a double overline the projection on the subspace orthogonal to $(\mathbf{I}-\hat{z}\otimes\hat{z})$ and $\hat{z}\otimes\hat{z}$. The first two terms in the RHS of Eq. (37) do not contribute to $\boldsymbol{\pi}_r^{[1]}$ but has to be retained for the next corrective terms. The heat flux term $\nabla\cdot\mathbf{q}_r$ is to be kept at this order when dealing with weakly nonlinear magnetosonic waves but arises at the next order when dealing with Alfvén waves. It is estimated in Sec. IV. When $\nabla\cdot\mathbf{q}_r$ is neglected, one recovers the classical gyroviscous tensor,[23–25]

$$\pi_{pxx}^{[1]}=-\pi_{pyy}^{[1]}=-\frac{p_{\perp p}}{2\Omega_p}(\partial_y u_x+\partial_x u_y), \tag{38}$$

$$\pi_{pzz}^{[1]}=0, \tag{39}$$

$$\pi^{[1]}_{pxy} = \pi^{[1]}_{pyx} = -\frac{p_{\perp p}}{2\Omega_p}(\partial_y u_y - \partial_x u_x), \tag{40}$$

$$\pi^{[1]}_{pyz} = \pi^{[1]}_{pzy} = \frac{1}{\Omega_p}[2p_{\parallel p}\partial_z u_x + p_{\perp p}(\partial_x u_z - \partial_z u_x)], \tag{41}$$

$$\pi^{[1]}_{pxz} = \pi^{[1]}_{pzx} = -\frac{1}{\Omega_p}[2p_{\parallel p}\partial_z u_y + p_{\perp p}(\partial_y u_z - \partial_z u_y)], \tag{42}$$

here given for the protons (the electron contribution being negligible due to the large mass ratio) and usually obtained in an $1/\Omega_p$ expansion.

The next correction $\boldsymbol{\pi}^{[2]}_p$ originates from terms neglected in Eq. (31), together with the dominant contributions in Eq. (32). We can consistently write (replacing single overlines by double ones)

$$\boldsymbol{\pi}^{[2]}_p \times \hat{z} - \hat{z} \times \boldsymbol{\pi}^{[2]}_p = \overline{\overline{L(\boldsymbol{\pi}^{[1]}_p)}} + \overline{\overline{\boldsymbol{\chi}^{[2]}_p}} + D[\boldsymbol{\chi}^{[1]}_p] + [(\hat{b} - \hat{z}) \times \boldsymbol{\pi}^{[1]}_p]^S, \tag{43}$$

together with the conditions

$$\hat{z} \cdot \pi^{[2]}_p \cdot \hat{z} + [(\hat{b} - \hat{z}) \cdot \pi^{[1]}_p \cdot \hat{z}]^S = 0, \quad \pi^{[2]}_p : \mathbf{I} = 0. \tag{44}$$

At the order of the present approximation,

$$\overline{\overline{L(\pi^{[1]}_p)}} = \frac{1}{\Omega_p}\partial_t \boldsymbol{\pi}^{[1]}_p. \tag{45}$$

Furthermore,

$$D[\boldsymbol{\chi}^{[1]}_p] \equiv \overline{\boldsymbol{\chi}^{[1]}_p} - \overline{\overline{\boldsymbol{\chi}^{[1]}_p}} = \tfrac{1}{2}[(\hat{b} - \hat{z}) \otimes \hat{z} + \hat{z} \otimes (\hat{b} - \hat{z})] : \boldsymbol{\chi}^{[1]}_p(\mathbf{I} - 3\hat{z} \otimes \hat{z}) + \tfrac{1}{2}(\mathbf{I} - 3\hat{z} \otimes \hat{z}) : \boldsymbol{\chi}^{[1]}_p[(\hat{b} - \hat{z}) \otimes \hat{z} + \hat{z} \otimes (\hat{b} - \hat{z})] \tag{46}$$

and

$$\boldsymbol{\chi}^{[2]}_p = \frac{1}{\Omega_p}\frac{d}{dt}(p_{\parallel p} - p_{\perp p})[(\hat{b} - \hat{z}) \otimes \hat{z}]^S + \frac{1}{\Omega_p}(p_{\parallel p} - p_{\perp p}) \times \{2(\hat{b} - \hat{z}) \cdot \nabla u \otimes \hat{z} + 2\hat{z} \cdot \nabla u \otimes (\hat{b} - \hat{z}) - 2(\hat{z} \cdot \nabla u \cdot \hat{z})(\hat{b} - \hat{z}) \otimes \hat{z} - [(\hat{b} - \hat{z}) \cdot \nabla u \cdot \hat{z} + \hat{z} \cdot \nabla u \cdot (\hat{b} - \hat{z})]\hat{z} \otimes \hat{z} + h \otimes \hat{z} - (h \cdot \hat{z})\hat{z} \otimes \hat{z}\}^S \tag{47}$$

with

$$h = \frac{1}{\Omega_p}\nabla \times \left[\frac{1}{4\pi\rho} b \times (\nabla \times b) - \frac{1}{\rho}\nabla \cdot \mathbf{P}^G_e\right]. \tag{48}$$

All the terms in $D[\boldsymbol{\chi}^{[1]}_p]$ and in $\boldsymbol{\chi}^{[2]}_p$ with the exception of those involving h (that originates from the generalized Ohm's law) result from field line distortion and are only relevant for the scaling of oblique Alfvén waves. For such waves, $\hat{z} \cdot (\hat{b} - \hat{z})$ is negligible, which enables one to write

$$\overline{\overline{\boldsymbol{\chi}^{[2]}_p}} + D[\boldsymbol{\chi}^{[1]}_p] = \frac{1}{\Omega_p}[(\hat{b} - \hat{z}) \cdot \nabla u \cdot \hat{z} + \hat{z} \cdot \nabla u \cdot (\hat{b} - \hat{z})][p_{\perp p}\mathbf{I} + (p_{\perp p} - 4p_{\parallel p})\hat{z} \otimes \hat{z}] + \frac{1}{\Omega_p}[p_{\perp p}\nabla \cdot u + (p_{\perp p} - 4p_{\parallel p})\hat{z} \cdot \nabla u \cdot \hat{z}] \times [(\hat{b} - \hat{z}) \otimes \hat{z}]^S + \frac{2}{\Omega_p}(p_{\perp p} - p_{\parallel p})[(\hat{b} - \hat{z}) \cdot \nabla u \otimes \hat{z} + \hat{z} \cdot \nabla u \otimes (\hat{b} - \hat{z})]^S + \frac{1}{\Omega_p}(p_{\perp p} - p_{\parallel p})(h \otimes \hat{z}). \tag{49}$$

This contribution is usually neglected,[26] and so are all the other terms in Eq. (43), except $(1/\Omega_p)(\partial/\partial t)\pi^{[1]}_p$. Retaining the nonlinear terms originating from the field line distortion is nevertheless important to prevent the onset of spurious nonlinearities (making the problem illposed) in the equation governing the dynamics of weakly nonlinear oblique Alfvén waves.[17] These waves appear to be governed by a *linear* Korteweg–de Vries equation with nonlocal damping. As in the Hall-MHD description, nonlinear couplings turn out to vanish.[27,28]

IV. MODELING OF THE HEAT FLUXES

It is again convenient in Eqs. (25) and (26), to separate the contributions to the gyrotropic part of $\nabla \cdot \mathbf{q}_r$, originating from the gyrotropic and nongyrotropic heat fluxes, by writing $\mathbf{q}_r = \mathbf{q}^G_r + \mathbf{q}^{NG}_r$ with

$$q^G_{r\,ijk} = q_{\parallel r}\hat{b}_i\hat{b}_j\hat{b}_k + q_{\perp r}(\delta_{ij}\hat{b}_k + \delta_{ik}\hat{b}_j + \delta_{jk}\hat{b}_i - 3\hat{b}_i\hat{b}_j\hat{b}_k). \tag{50}$$

The equations for the gyrotropic pressure components involve

$$\hat{b} \cdot (\nabla \cdot \mathbf{q}^G_r) \cdot \hat{b} = \nabla \cdot (\hat{b} q_{\parallel r}) - 2q_{\perp r}\nabla \cdot \hat{b}, \tag{51}$$

$$\tfrac{1}{2}[\mathrm{tr}(\nabla \cdot \mathbf{q}^G_r) - \hat{b} \cdot (\nabla \cdot \mathbf{q}^G_r) \cdot \hat{b}] = \nabla \cdot (\hat{b} q_{\perp r}) + q_{\perp r}\nabla \cdot \hat{b}, \tag{52}$$

together with the contribution of the nongyrotropic heat fluxes to the gyrotropic part of $\nabla \cdot \mathbf{q}_r$ that we denote $(\nabla \cdot \mathbf{q}^{NG}_r)^G$. The nongyrotropic part of $\nabla \cdot \mathbf{q}_r$ contributes to the nongyrotropic pressure corrections.

A. Nongyrotropic heat flux contributions

The gyrotropic heat flux contributions are comparable to the pressure perturbations (as seen from the gyrotropic pressure equations), i.e., of order $\epsilon^{1/2}$ for parallel Alfvén waves and of order ϵ for oblique Alfvén and magnetosonic waves. On the other hand, the nongyrotropic heat flux components do not only behave like the product of a pressure and a velocity but also involve an additional space derivative arising together with the $1/\Omega_r$ factor. From the scaling assumptions, one can conclude that these contributions are subdominant for both parallel Alfvén and oblique magnetosonic waves, while they are of the same order as the gyrotropic heat flux components in the case of oblique Alfvén waves. This observation is confirmed by the kinetic theory based on Vlasov–Maxwell equations (see Refs. 16 and 17 and Appendix B).

The nongyrotropic heat fluxes obtained for oblique Alfvén waves (Appendices B and C) can be expressed in terms of current $j=(c/4\pi)\nabla\times b$ and diamagnetic drifts of each particle species $u_{d,r}=(c/nq|b|^2)b\times\nabla\cdot\mathbf{p}_r$ that in the considered limit are given by $j/qn\approx v_A^2/\Omega_p$ $(-\partial_z(b_y/B_0),0,\partial_x(b_y/B_0))$ and $u_{d,r}\approx v_{\Delta r}^2/\Omega_p(\partial_z(b_y/B_0),0,0)$, where we define the squared Alfvén velocity $v_A^2=B_0^2/4\pi\rho^{(0)}$ and also $v_{\Delta r}^2=(p_{\perp r}^{(0)}-p_{\| r}^{(0)})/\rho^{(0)}$. We thus infer the closure approximation

$$(\nabla\cdot\mathbf{q}_e^{NG})^G=2\nabla_\perp\cdot\left[p_{\perp e}\left(u_{d,e}-\frac{j}{qn}\right)\right](I-\hat{b}\otimes\hat{b})+\nabla_\perp\cdot\left[p_{\| e}\left(u_{d,e}-\frac{j}{qn}\right)\right]\hat{b}\otimes\hat{b},\tag{53}$$

$$(\nabla\cdot\mathbf{q}_p^{NG})^G=2\nabla_\perp\cdot[p_{\| p}u_{d,p}]\hat{b}\otimes\hat{b},\tag{54}$$

and also approximate the nongyrotropic part of $\nabla\cdot\mathbf{q}_p$ by

$$\overline{\overline{\nabla\cdot\mathbf{q}}}_p=\frac{p_{\perp p}}{2}[\nabla_\perp\otimes u_{dp}-(\hat{z}\times\nabla_\perp)\otimes(\hat{z}\times u_{dp})]+\left\{\hat{z}\otimes\left[\nabla_\perp q_{\perp p}-p_{\perp p}\frac{v_{\Delta p}^2}{2\Omega_p}\hat{z}\times\Delta_\perp\hat{b}-2p_{\| p}\hat{z}\times(\nabla\times u_{dp})\right]\right\}^S.\tag{55}$$

B. Gyrotropic heat fluxes

In order to infer closed expressions for the gyrotropic heat fluxes, based on the predictions of the kinetic theory for oblique Alfvén waves (Appendix C), we adapt the approach developed in the context of parallel propagation[12] where the closure approximations eventually reduce to the replacement of the plasma response function W_r by its two-or four-pole Padé approximants.

1. Parallel heat flux

In order to reduce the problem to a form close to that of parallel propagation, starting from Eq. (C11) where quantities proportional to m_e/m_p have been neglected, we first define

$$\frac{q'_{\| r}}{v_{th,r}p_{\| r}^{(0)}}=\frac{q_{\| r}}{v_{th,r}p_{\| r}^{(0)}}-3\left(\frac{v_{\Delta e}^2+v_A^2}{v_A^2}\right)\left(\frac{\Omega_p}{\Omega_r}-1\right)\frac{j_\|}{nqv_{th,r}},\tag{56}$$

where we used the relation $c_r^2=(v_A^2+v_{\Delta e}^2+v_{\Delta p}^2)/v_{th,r}^2$ and the expression of the parallel current in the asymptotics of long oblique Alfvén waves given in the preceding section. It follows that:

$$\frac{q'_{\| r}}{v_{th,r}p_{\| r}^{(0)}}=c_r\frac{c_r^2-3+W_r^{-1}}{c_r^2-1+W_r^{-1}}\frac{T_{\| r}^{(1)}}{T_{\| r}^{(0)}},\tag{57}$$

where $T_{\| r}^{(1)}$ denotes the parallel temperature perturbations for the particles of species r and $T_{\| r}^{(0)}$ the corresponding equilibrium value. Similar notations are used for the transverse temperatures. We then proceed as in Ref. 16. The parameter c_r defined as the ratio of the phase velocity projected on the direction of the ambient field to the thermal velocity of species r is now more generally viewed as the ratio $X=-(1/v_{th,r})\partial_t\partial_z^{-1}$. The operator $\mathcal{F}_\|(X)$ defined by

$$\frac{q'_{\| r}}{v_{th,r}p_{\| r}^{(0)}}=\mathcal{F}_\|(X)\frac{T_{\| r}^{(1)}}{T_{\| r}^{(0)}}\tag{58}$$

is approximated by a homographic function

$$\mathcal{F}_\|(X)=(Q_\|^3+Q_\|^4X\mathcal{H})^{-1}(Q_\|^1X+Q_\|^2\mathcal{H}),\tag{59}$$

where $\mathcal{H}$ is the Hilbert transform with respect to the parallel coordinate z, which allows one to eventually get a first-order initial value problem. The constant coefficients $Q_\|^i$ are chosen in a way that ensures the correct asymptotic behavior of the parallel heat fluxes in both the isothermal and adiabatic limits. As shown by SHD, this prescription results in a satisfactory modeling in the intermediate regimes. In the isothermal limit $(c_r\ll 1)$, $\mathcal{W}_r\approx 1-c_r^2+\sqrt{\pi/2}c_r\mathcal{H}_\xi$ and $q_{r,\| r}^{(1)}=-\sqrt{8/\pi}v_{th,r}n^{(0)}\mathcal{H}_\xi T_{\| r}^{(1)}$ independent of c_r. Differently, in the adiabatic limit $(c_r\gg 1)$, $W_r\approx -1/c_r^2-3/c_r^4-15/c_r^6$ and the heat fluxes are negligible. One thus gets $Q_\|^1=0$, $Q_\|^2=-\sqrt{8/\pi}$, $Q_\|^3=1$, $Q_\|^4=-\sqrt{8/\pi}(3\pi/8-1)$. In this approximation, the corrected parallel heat flux $q'_{\| r}$ is thus determined in terms of the parallel temperature $T_{\| r}$ by the partial differential equation

$$\left(\frac{d}{dt}+\frac{v_{th,r}}{\sqrt{\frac{8}{\pi}}\left(1-\frac{3\pi}{8}\right)}\mathcal{H}\partial_z\right)\frac{q'_{\| r}}{v_{th,r}p_{\| r}^{(0)}}=\frac{1}{1-\frac{3\pi}{8}}v_{th,r}\partial_z\frac{T_{\| r}}{T_{\| r}^{(0)}},\tag{60}$$

where, to restore Galilean invariance, the convective derivative $\partial_t+u\cdot\nabla$ has been substituted to the partial time derivative.

2. Perpendicular heat flux

Proceeding in a similar way, starting from Eq. (C12) or (C13), we first define

$$\frac{q^{\star}_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} = \frac{q_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} + \left[\left(1+\frac{v^2_{\Delta e}+v^2_{\Delta p}}{v_A^2}\right)\left(\frac{\Omega_p}{\Omega_r}+1\right) - \frac{v^2_{\Delta p}}{v_A^2} + 2\frac{v^2_{\Delta r}}{v_A^2}\frac{\Omega_p}{\Omega_r}\right]\frac{j_\parallel}{nqv_{th,r}} \quad (61)$$

that, for long oblique Alfvén waves, can be expressed either in terms of $A=(b_z/B_0)+(b_y^2/2B_0^2)$ or in terms of $(T^{(1)}_{\perp r}/T^{(0)}_{\perp r}) -(3/v_{th,r})\partial_t\partial_z^{-1}(\Omega_p/\Omega_r)(v^2_{th,r}/v_A^2)(j_\parallel/nqv_{th,r})$, by means of operators that as previously are to be approximated. In order to accurately fit the adiabatic and isothermal limits, it is convenient to use a mixed expression involving both dependencies, in the form

$$\frac{q^{\star}_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} = \mathcal{F}^1_\perp\left(-\frac{1}{v_{th,r}}\partial_t\partial_z^{-1}\right) \times\left[\frac{T^{(1)}_{\perp r}}{T^{(0)}_{\perp r}} - \frac{3}{v_{th,r}}\partial_t\partial_z^{-1}\frac{\Omega_p}{\Omega_r}\frac{v^2_{th,r}}{v_A^2}\frac{j_\parallel}{nqv_{th,r}}\right] + \mathcal{F}^2_\perp\left(-\frac{1}{v_{th,r}}\partial_t\partial_z^{-1}\right)A. \quad (62)$$

Prescribing again a homographic form for the operators

$$\mathcal{F}^1_\perp(X) = (Q^3_\perp + Q^4_\perp X\mathcal{H})^{-1}(Q^1_\perp X + Q^2_\perp\mathcal{H}), \quad (63)$$

$$\mathcal{F}^2_\perp(X) = (Q^3_\perp + Q^4_\perp X\mathcal{H})^{-1}(Q^5_\perp X + Q^6_\perp\mathcal{H}), \quad (64)$$

we are led to choose $Q^1_\perp=Q^5_\perp=0$, $Q^2_\perp=Q^4_\perp=-\sqrt{2/\pi}$, and $Q^6_\perp=\sqrt{2/\pi}(1-T^{(0)}_{\perp r}/T^{(0)}_{\parallel r})$ and get

$$\left(\frac{\partial}{\partial t} - \sqrt{(\pi/2)}v_{th,r}\mathcal{H}\partial_z\right)\frac{q^{\star}_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} = v_{th,r}\partial_z\left[\left(1-\frac{T^{(0)}_{\perp r}}{T^{(0)}_{\parallel r}}\right)\frac{|b|}{B_0} - \left(\frac{T_{\perp r}}{T^{(0)}_{\perp r}} - \frac{3v_{th,r}}{v_A^2}\frac{\Omega_p}{\Omega_r}\partial_t\partial_z^{-1}\frac{j_\parallel}{nqv_{th,r}}\right)\right]. \quad (65)$$

Introducing

$$\frac{q'_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} = \frac{q^{\star}_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} - 3\frac{v^2_{th,r}}{v_A^2}\frac{\Omega_p}{\Omega_r}\frac{j_\parallel}{nqv_{th,r}} = \frac{q_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} + \left[\left(1+\frac{v^2_{\Delta e}+v^2_{\Delta p}}{v_A^2}\right)\left(\frac{\Omega_p}{\Omega_r}+1\right) - \frac{v^2_{\Delta p}}{v_A^2} + \frac{2v^2_{\Delta r}-3v^2_{th,r}}{v_A^2}\frac{\Omega_p}{\Omega_r}\right]\frac{j_\parallel}{nqv_{th,r}}, \quad (66)$$

we finally obtain

$$\left(\frac{d}{dt} - \sqrt{\frac{\pi}{2}}v_{th,r}\mathcal{H}\partial_z\right)\frac{q'_{\perp r}}{v_{th,r}p^{(0)}_{\perp r}} = v_{th,r}\partial_z\left[\left(1-\frac{T^{(0)}_{\perp r}}{T^{(0)}_{\parallel r}}\right)\frac{|b|}{B_0} - \frac{T_{\perp r}}{T^{(0)}_{\perp r}} + 3\sqrt{\frac{\pi}{2}}\frac{v^2_{th,r}}{v_A^2}\frac{\Omega_p}{\Omega_r}\mathcal{H}\frac{j_\parallel}{nqv_{th,r}}\right]. \quad (67)$$

As previously, a convective derivative has been substituted to the partial time derivatives in order to restore Galilean invariance.

V. THE MODEL AND ITS VALIDATION

A monofluid model has thus been constructed. It is defined by the closed system formed by Eqs. (9), (10), (16), (25), and (26) where the s_{ir} terms are neglected and where Eqs. (51)–(54) have been used, supplemented by Eqs. (56), (60), (66), and (67), together with the nongyrotropic pressure corrections $\boldsymbol{\pi}_r=\boldsymbol{\pi}_r^{[1]}+\boldsymbol{\pi}_r^{[2]}$ that are computed in Sec. III B and involve the nongyrotropic heat flux given by Eq. (55).

To validate this model, we consider its predictions for the various MHD waves in the long-wavelength limit. Our previous model[12] specifically designed to describe parallel Alfvén waves is easily recovered by prescribing the ordering associated to these waves. The nongyrotropic heat flux contributions then disappear and only the leading order gyroviscous tensor without the heat flux term is to be retained. In this regime, a reductive perturbative expansion leads to a generalized kinetic derivative nonlinear Schrödinger equation that identifies with that derived from the Vlasov–Maxwell system,[16] up to the replacement of the plasma response function by its two- or four-pole Padé approximants.[12]

The model derived in the present paper is in contrast needed to describe oblique Alfvén waves. We demonstrate in this section that the kinetic theory presented in Appendix C is accurately reproduced, and so are the classical dispersion relations and Landau damping rates of oblique and kinetic Alfvén waves.

Denoting by ξ the coordinate along the direction of propagation, one has $\boldsymbol{\nabla}=(\sin\alpha\,\partial_\xi, 0, \cos\alpha\,\partial_\xi)$ and $\partial_t=-V_0\partial_\xi$ with $V_0=\Lambda_0\cos\alpha$. From Eq. (60), one immediately gets to leading order

$$\frac{q'_{\parallel r}}{v_{th,r}p^{(0)}_{\parallel r}} = \frac{-\sqrt{\frac{8}{\pi}}\mathcal{H}}{1-\sqrt{\frac{8}{\pi}}\left(\frac{3\pi}{8}-1\right)c_r\mathcal{H}}\frac{T^{(1)}_{\parallel r}}{T^{(0)}_{\parallel r}} = c_r\frac{c_r^2-3+W_{4r}^{-1}}{c_r^2-1+W_{4r}^{-1}}\frac{T^{(1)}_{\parallel r}}{T^{(0)}_{\parallel r}}, \quad (68)$$

which for the protons and the electrons, respectively, give

$$\frac{q^{(1)}_{\parallel p}}{v_{th,p}p^{(0)}_{\parallel p}} = c_p\frac{c_p^2-3+W_{4p}^{-1}}{c_p^2-1+W_{4p}^{-1}}\frac{T^{(1)}_{\parallel p}}{T^{(0)}_{\parallel p}}, \quad (69)$$

$$\frac{q^{(1)}_{\parallel e}}{v_{th,p}p^{(0)}_{\parallel p}} = c_e\frac{c_e^2-3+W_{4e}^{-1}}{c_e^2-1+W_{4e}^{-1}}\frac{T^{(1)}_{\parallel e}}{T^{(0)}_{\parallel e}} - 3\frac{v_A^2+v^2_{\Delta e}}{v_{th,e}}\frac{\sin\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0}. \quad (70)$$

This reproduces Eq. (C11), up to the replacement of the plasma response function $\mathcal{W}_r$ by its four-pole approximant defined as

$$\mathcal{W}_{4r} = \frac{\frac{1}{2}(8-3\pi)c_r^2 - \sqrt{2\pi}c_r\mathcal{H} + 4}{\frac{1}{2}c_r^4(3\pi-8) + \sqrt{2\pi}c_r^3\mathcal{H} + \frac{1}{2}(16-9\pi)c_r^2 - 3\sqrt{2\pi}c_r\mathcal{H} + 4}. \tag{71}$$

Substituting in the equation for the parallel proton pressure that rewrites

$$\frac{p_{\parallel p}^{(1)}}{p_{\parallel p}^{(0)}} - 3\frac{n^{(1)}}{n^{(0)}} + 2A = \frac{2}{\Lambda_0}\frac{v_A^2 + v_{\Delta e}^2 + v_{\Delta p}^2}{\Omega_p}\sin\alpha\partial_\xi\frac{b_y^{(1)}}{B_0} + \frac{1}{c_r}\frac{q_{\parallel p}^{(1)}}{v_{th,p}p_{\parallel p}^{(0)}}, \tag{72}$$

one gets

$$\frac{p_{\parallel p}^{(1)}}{p_{\parallel p}^{(0)}} = (c_p^2 + \mathcal{W}_{4p}^{-1})\frac{n^{(1)}}{n^{(0)}} - (c_p^2 - 1 + \mathcal{W}_{4p}^{-1})A + (c_p^2 - 1 + \mathcal{W}_{4p}^{-1})\frac{\Lambda_0}{\Omega_p}\sin\alpha\partial_\xi\frac{b_y^{(1)}}{B_0}, \tag{73}$$

$$\frac{p_{\parallel e}^{(1)}}{p_{\parallel e}^{(0)}} - 3\frac{n^{(1)}}{n^{(0)}} + 2A = \frac{c_e^2 - 3 + \mathcal{W}_{4e}^{-1}}{c_e^2 - 1 + \mathcal{W}_{4e}^{-1}}\left(\frac{p_{\parallel e}^{(1)}}{p_{\parallel e}^{(0)}} - \frac{n^{(1)}}{n^{(0)}}\right) \tag{74}$$

that correspond to Eq. (C6).

From the transverse heat flux equations, we get

$$\frac{q_{\perp e}^{(1)}}{v_{th,e}p_{\perp e}^{(0)}} = \frac{-1}{c_e + \sqrt{\frac{\pi}{2}}\mathcal{H}}\left[\left(1 - \frac{T_{\perp e}^{(0)}}{T_{\parallel e}^{(0)}}\right)A - \frac{p_{\perp e}^{(1)}}{p_{\perp e}^{(0)}} + \frac{n^{(1)}}{n^{(0)}}\right] - \frac{v_A^2 + v_{\Delta e}^2}{v_{th,e}}\frac{\sin\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0}. \tag{75}$$

Substituting in the equation for the electron parallel pressure that rewrites

$$\frac{p_{\perp e}^{(1)}}{p_{\perp e}^{(0)}} = \frac{n^{(1)}}{n^{(0)}} + A + \frac{v_A^2 + v_{\Delta e}^2}{\Lambda_0}\frac{\sin\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0} + \frac{q_{\perp e}^{(1)}}{\Lambda_0 p_{\perp e}}, \tag{76}$$

one obtains

$$\frac{p_{\perp e}^{(1)}}{p_{\perp e}^{(0)}} = \frac{n^{(1)}}{n^{(0)}} + \left(1 - \frac{T_{\parallel e}^{(0)}}{T_{\parallel e}^{(0)}}\mathcal{W}_{2e}\right)A. \tag{77}$$

Here,

$$\mathcal{W}_{2r} = \frac{1}{1 - \sqrt{\frac{\pi}{2}}c_r\mathcal{H} - c_r^2} \tag{78}$$

is the two-pole approximant of the plasma response function $\mathcal{W}_r$.

Similarly, for the protons

$$\frac{p_{\perp p}^{(1)}}{p_{\perp p}^{(0)}} = \frac{n^{(1)}}{n^{(0)}} + A - \frac{v_A^2 + v_{\Delta e}^2}{\Lambda_0}\frac{\sin\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0} + \frac{q_{\perp p}^{(1)}}{\Lambda_0 p_{\perp p}} \tag{79}$$

with

$$\frac{q_{\perp p}^{(1)}}{v_{th,p}p_{\perp p}^{(0)}} = \frac{-1}{c_p + \sqrt{\frac{\pi}{2}}\mathcal{H}}\left[\left(1 - \frac{T_{\perp p}^{(0)}}{T_{\parallel p}^{(0)}}\right)A - \frac{p_{\perp p}^{(1)}}{p_{\perp p}^{(0)}} + \frac{n^{(1)}}{n^{(0)}} + 3\sqrt{\frac{\pi}{2}}v_{th,p}\mathcal{H}\frac{\sin\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0}\right] + [3(v_{th,p}^2 - v_{\Delta p}^2) - 2(v_A^2 + v_{\Delta e}^2)]\frac{\sin\alpha}{v_{th,p}\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0}, \tag{80}$$

which implies

$$\frac{p_{\perp p}^{(1)}}{p_{\perp p}^{(0)}} = \frac{n^{(1)}}{n^{(0)}} + \left(1 - \frac{T_{\perp p}^{(0)}}{T_{\parallel p}^{(0)}}W_{2p}\right)A - 3\Lambda_0\frac{\sin\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0}. \tag{81}$$

Again the result of the kinetic theory, as given by Eq. (C8), is recovered up to the replacement of the plasma response function by a Padé approximant.

To push further the validation of the present model, it is of interest to concentrate on the regime $(m_e/m_i) \ll \beta \ll (T_e/T_p)$, with $\beta = (1/v_A^2)(T_e/m_p)$, assuming no temperature anisotropy for easier comparison with classical results. This ordering corresponds to the limit $c_e \to 0$ of isothermal electrons and $c_p \to \infty$ of adiabatic protons.

In the limit $c_e \to 0$, $W_{4e} \approx W_{2e} \approx W_e \approx 1 - c_e^2 + \sqrt{\pi/2}c_e\mathcal{H}$ and we get

$$\frac{p_{\parallel e}^{(1)}}{\rho^{(0)}} = \beta v_A^2\frac{p_{\parallel e}^{(1)}}{p_{\parallel e}^{(0)}} = v_A^2\left\{\left[\beta - \sqrt{\frac{\pi}{2}}\sqrt{\beta}\sqrt{\frac{m_e}{m_p}}\mathcal{H}\right]\frac{\rho^{(1)}}{\rho^{(0)}} + \sqrt{\beta}\sqrt{\frac{\pi}{2}}\sqrt{\frac{m_e}{m_p}}\mathcal{H}A\right\}, \tag{82}$$

$$\frac{p_{\perp e}^{(1)}}{\rho^{(0)}} = \beta v_A^2\frac{p_{\perp e}^{(1)}}{p_{\perp e}^{(0)}} = v_A^2\left[\beta\frac{\rho^{(1)}}{\rho^{(0)}} - \sqrt{\beta}\sqrt{\frac{\pi}{2}}\sqrt{\frac{m_e}{m_p}}\mathcal{H}A\right], \tag{83}$$

which provides a systematic derivation of relations previously based on a phenomenological argument.[17]

The adiabatic limit $c_p \to \infty$ assumes that the phase velocity of the wave is much larger than the thermal velocity, which is not consistent with the long-wave asymptotics. The adiabatic limit is thus conveniently taken by prescribing zero heat fluxes, and the relations (60) and (61) of Ref. 17 are then immediately recovered. By inspection, it is also easily verified that the gyroviscous tensor defined by Eq. (43) identifies within the reductive perturbative scaling with Eqs. (B2)–(B7) of Ref. 17. In particular Eq. (49) reproduces Eqs.

(B15)–(B18) of the same reference. The remaining of the asymptotic analysis is straightforward and is performed in Ref. 17. Direct comparisons are successfully made with kinetic results.[18,19] This demonstrates that the present model correctly reproduces the dynamics of small amplitude oblique and kinetic Alfvén waves.

The case of oblique magnetosonic waves requires a more detailed discussion that we explicit in the regime of adiabatic ions and isothermal electrons with isotropic temperatures. The leading order linear dispersion relation correctly reproduces that provided by the kinetic theory [see Eqs. (29) and (A21) of Ref. 17]. It includes a Landau damping rate that, up to an angular factor, scales like $kv_A\sqrt{\beta}\sqrt{(m_e/m_p)}$ where, as above, β is defined as the ratio of the electron to magnetic pressures and k the wave number of the perturbation. This level of description is sufficient when this rate of damping is larger than the inverse nonlinear time ku (where u is a typical velocity perturbation), that is to say when $\sqrt{\beta}\sqrt{(m_e/m_p)} \gg \epsilon(V_0/v_A)$. The parameter ϵ can be estimated as kl_p, where $l_p=(v_A/\Omega_p)$ is the proton inertial length. Let us first consider the distinguished limit where the wave amplitude scales like ϵ. For slow waves for which $V_0 \sim \sqrt{\beta}v_A$, the condition reduces to $kl_p \ll (m_e/m_p)^{1/4} \approx 0.15$. These waves are thus strongly damped in the long-wave limit. For fast waves $V_0 \sim v_A$, and the condition for rapid damping reads $\beta \gg (m_p/m_e)(kl_p)^4$. When this condition is not satisfied, the Landau damping arises at the same order as the nonlinear and dispersive terms and a weakly nonlinear analysis on the time scale $\tau=\epsilon^{3/2}$ is required. In this regime, the equations for $\partial_\tau u_x^{(1)}$ and $\partial_\tau u_z^{(1)}$ involve the quantities $\partial_x p_\perp^{(2)}$ and $\partial_z p_\parallel^{(2)}$, and thus the gyrotropic heat fluxes $q_\perp^{(2)}$ and $q_\parallel^{(2)}$, together with the FLR term $\boldsymbol{\pi}_r^{[2]}$ that through Eq. (43) is prescribed by $1/\Omega_p(\partial_t \boldsymbol{\pi}_r^{[1]} + \overline{\overline{\boldsymbol{\nabla}\cdot\mathbf{q}_p^{(3/2)}}})$. These heat fluxes, when not negligible, are not properly modeled in the present formalism. They are absent in the case of purely transverse propagation, a situation addressed in Ref. 29. The case of oblique propagation in the adiabatic limit was considered in Ref. 30 where the term $(1/\Omega_p)\partial_t \boldsymbol{\pi}_r^{[1]}$ was overlooked.

When the amplitude is larger, usual MHD supplemented by $1/\Omega_p$ corrections provides a sufficient description. As the amplitude of these waves is reduced by dissipation, the regime dominated by Landau damping is recovered. The only case where our model does not provide a complete description of magnetosonic waves thus concerns small amplitude waves with the distinguished scaling and very small β.

The question arises whether the usual energy $E = \int[(\rho(u^2/2)+(b^2/8\pi)+p_\perp+(1/2)p_\parallel)]d^3x$ is conserved by the above monofluid model. The delicate contributions originate from the electron pressure gradient in the induction equation and from the second-order nongyrotropic pressure corrections. The first term that affects the magnetic field evolution only in the case of pressure anisotropy contributes in a long wave theory at the level of the linear dispersion relation. In this limit, it can thus be replaced by $(1/\rho_0)\boldsymbol{\nabla} p_{\perp e} - (v_{\Delta e}^2/B_0^2)\boldsymbol{\nabla}\cdot(b\otimes b)$, a term that does not contribute to the energy budget. Concerning the nongyrotropic pressure contributions, while the leading order $\boldsymbol{\pi}^{[1]}$ preserves energy (at least in the absence of the heat flux term), the effect of $\boldsymbol{\pi}^{[2]}$ is still unsettled. This question requires further investigations. In fact, in a way similar to the diamagnetic term in the generalized Ohm's law, this contribution is only relevant at the level of the linear dispersion relation of oblique and kinetic Alfvén waves. As a consequence, even in the case where the energy is only conserved at the order of validity of the performed approximations, the effect on the large-scale dynamics will be negligible.

VI. CONCLUSION

A monofluid model has been derived with the constraint to reproduce the weakly nonlinear dynamics and the Landau damping of long MHD waves in a collisionless plasma, for any β larger than the electron to proton mass ratio and any angle of propagation. It reproduces the dynamics of small-amplitude oblique Alfvén waves, including the exact cancellation of the nonlinearity. For parallel Alfvén waves, it leads to the KDNLS equation and describes the transverse instability of a circularly polarized wave,[31] resulting in the formation of intense magnetic filaments. This Alfvén wave "collapse" was considered as a possible mechanism at the origin of the cylindrical field aligned current tubes observed by the CLUSTER mission in the terrestrial magnetosheath.[32]

Comparison of the model with gyrokinetic simulations and possibly with Vlasov–Maxwell or particles in cells simulations, in particular, in the nonlinear stage of parametric instabilities, are in project. It is also necessary to evaluate the importance of particle trapping that requires a nonlinear fluid closure, presently very difficult to design.[33]

This model can be used to perform three-dimensional numerical simulations of dispersive MHD turbulence, taking into account realistic dissipation and heating mechanisms. The retained second-order FLR corrections should in particular provide an accurate description of the kinetic Alfvén waves generated at small scales. Such simulations, that involve a self-consistent treatment of the turbulent dynamics in the presence of Landau damping, could significantly contribute to the understanding of cosmic ray scattering (by fast and Alfvén modes) in the interstellar medium.[34]

This model will also be useful to study the formation of coherent structures such as magnetic holes,[35,36] shocklets, and also the structures resulting from the nonlinear evolution of the mirror instability, observed in the solar wind[37] and the magnetosheath.[38] A correct description of this instability that extends up to scales comparable to the ion Larmor radius requires higher order FLR corrections that, as mentioned in Section III B, cannot be directly obtained within a monofluid description. Their evaluation is possible through a $1/\Omega_p$ expansion of all the fields in a multifluid description.[39]

Another development concerns hybrid simulations that could possibly be improved by replacing the usual MHD description of the electron dynamics by a more refined one including physical processes retained by the present model.

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APPENDIX A: LONG-WAVE EXPANSION OF VLASOV–MAXWELL EQUATIONS FOR OBLIQUE ALFVÉN WAVES

We write the Vlasov–Maxwell equations in the form

$$\partial_t f_r + v \cdot \nabla f_r + \frac{q_r}{m_r}\left(e + \frac{1}{c} v \times b\right) \cdot \nabla_v f_r = 0, \tag{A1}$$

$$\frac{1}{c}\partial_t b = - \nabla \times e, \tag{A2}$$

$$\nabla \times b = \frac{4\pi}{c}\sum_r q_r n_r \int v f_r d^3v + \frac{1}{c}\partial_t e, \tag{A3}$$

$$\nabla \cdot e = 4\pi \sum_r q_r n_r \int f_r d^3 v, \tag{A4}$$

where f_r and n_r are the distribution function and the average number density of the particles of species r with charge q_r and mass m_r. The displacement current $(1/c)\partial_t e$ turns out to be negligible in the present analysis where perturbations propagate at a velocity small compared with the speed of light. This contribution which might be important for auroral plasmas is retained by Verheest.[40]

Let α be the angle between the ambient magnetic field $B_0\hat{z}$ (where $\hat{z}$ is the unit vector pointing along the zaxis) and the direction of propagation of the wave. It is then convenient to perform the change of frame $x' = x \cos \alpha - z \sin \alpha$, $z' = x \sin \alpha + z \cos \alpha$, the dynamics being assumed independent of the y variable. We then introduce the stretched variable $\xi = \epsilon^{1/2}(z' - V_0 t)$ where $V_0 \ll c$ is the Alfvén-wave propagation velocity in the z direction, together with the slow time $\tau = \epsilon^{3/2} t$. It follows that the spatial gradient rewrites $\nabla = (\epsilon^{1/2} \sin \alpha \partial_\xi, 0, \epsilon^{1/2} \cos \alpha \partial_\xi)$.

In order to select oblique Alfvén waves, we expand

$$b_x = \epsilon(b_x^{(1)} + \epsilon b_x^{(2)} + \cdots), \tag{A5}$$

$$b_y = \epsilon^{1/2}(b_y^{(1)} + \epsilon b_y^{(2)} + \cdots), \tag{A6}$$

$$b_z = B_0 + \epsilon(b_z^{(1)} + \epsilon b_z^{(2)} + \cdots), \tag{A7}$$

and thus, from Eq. (A2),

$$e_x = \epsilon^{1/2}(e_x^{(1)} + \epsilon e_x^{(2)} + \cdots), \tag{A8}$$

$$e_y = \epsilon(e_y^{(1)} + \epsilon e_y^{(2)} + \cdots), \tag{A9}$$

$$e_z = \epsilon^{1/2}(e_z^{(1)} + \epsilon e_z^{(2)} + \cdots), \tag{A10}$$

with

$$\frac{V_0}{c} b_x^{(1)} = - \cos \alpha e_y^{(1)}, \tag{A11}$$

$$\frac{V_0}{c} b_y^{(1)} = \cos \alpha e_x^{(1)} - \sin \alpha e_z^{(1)}, \tag{A12}$$

$$\frac{V_0}{c} b_z^{(1)} = \sin \alpha e_y^{(1)}. \tag{A13}$$

We also expand the distribution function in the form

$$f_r = F_r^{(0)} + \epsilon^{1/2} f_r^{(0)} + \epsilon f_r^{(1)} + \epsilon^{3/2} f_r^{(2)} + \cdots, \tag{A14}$$

where $F_r^{(0)}$ denotes the equilibrium velocity distribution function, assumed rotationally symmetric around the direction of the ambient field and symmetric relatively to forward and backward velocities along this direction, thus excluding the presence of equilibrium drifts.[40–42]

It is also convenient to express the velocity v in a cylindrical coordinate system by defining the azimuthal angle $\phi = \tan^{-1}(v_z/v_y)$ of the velocity component transverse to the ambient magnetic field. One writes

$$v = (v_x = v_\perp \cos \phi, v_y = v_\perp \sin \phi, v_z = v_\parallel) \tag{A15}$$

and

$$\nabla_v = \left(\cos \phi\, \partial_{v_\perp} - \frac{\sin \phi}{v_\perp}\partial_\phi, \sin \phi\, \partial_{v_\perp} + \frac{\cos \phi}{v_\perp}\partial_\phi, \partial_{v_\parallel}\right). \tag{A16}$$

Furthermore $(q_r/cm_r)(v \times B_0 \hat{z}) \cdot \nabla_v = -\Omega_r \partial_\phi$, where $\Omega_r = (q_r B_0)/(m_r c)$ is the gyrofrequency of the particles of species r.

Expanding to the successive orders, one gets from Eq. (A1),

$$\Omega_r \partial_\phi F_r^{(0)} = 0, \tag{A17}$$

$$\Omega_r \partial_\phi f_r^{(0)} = \frac{q_r}{m_r}\Sigma_1^{(1)} F_r^{(0)}, \tag{A18}$$

$$\Omega_r \partial_\phi f_r^{(1)} = \frac{q_r}{m_r}(\Sigma_2^{(1)} F_r^{(0)} + \Sigma_1^{(1)} f_r^{(0)}) + \Sigma_3 f_r^{(0)}, \tag{A19}$$

$$\Omega_r \partial_\phi f_r^{(2)} = \frac{q_r}{m_r}(\Sigma_1^{(2)} F_r^{(0)} + \Sigma_2^{(1)} f_r^{(0)} + \Sigma_1^{(1)} f_r^{(1)}) + \Sigma_3 f_r^{(1)}, \tag{A20}$$

where

$$\Sigma_1^{(s)} = \left(e_x^{(s)} - \frac{v_z}{c} b_y^{(s)}\right)\left(\cos \phi\, \partial_{v_\perp} - \frac{\sin \phi}{v_\perp}\partial_\phi\right) + \left(e_z^{(s)} + \frac{v_x}{c} b_y^{(s)}\right)\partial_{v_\parallel}, \tag{A21}$$

$$\Sigma_2^{(s)} = \frac{v_y}{c} b_z^{(s)}\left(\cos \phi\, \partial_{v_\perp} - \frac{\sin \phi}{v_\perp}\partial_\phi\right) + \left(e_y^{(s)} + \frac{v_z b_x^{(s)} - v_x b_z^{(s)}}{c}\right)\left(\sin \phi\, \partial_{v_\perp} + \frac{\cos \phi}{v_\perp}\partial_\phi\right) - \frac{v_y b_x^{(s)}}{c}\partial_{v_\parallel}, \tag{A22}$$

$$\Sigma_3=(v_x\sin\alpha+v_z\cos\alpha-V_0)\partial_\xi. \quad (A23)$$

Equation (A17) indicates that $F_r^{(0)}$ is independent of the angle ϕ. The solvability of (A18)) implies $e_z^{(1)}=0$ and by (A12), $e_x^{(1)}=(\Lambda_0/c)b_y^{(1)}$ where $\Lambda_0=V_0/\cos\alpha$. This equation is then solved as

$$f_r^{(0)}=\mathcal{D}F_r^{(0)}\sin\phi\frac{b_y^{(1)}}{B_0} \quad (A24)$$

with $\mathcal{D}=(\Lambda_0-v_\parallel)\partial_{v_\perp}+v_\perp\partial_{v_\parallel}$. We also used the solvability condition of Eq. (A19), that reads $\overline{f_r^{(0)}}=\langle f_r^{(0)}\rangle\equiv 1/2\pi\int f_r^{(0)}d\phi=0$.

It follows from the sin ϕ dependence of $f_r^{(0)}$ that $\partial_\phi f_r^{(1)}$ only contains sin ϕ and sin 2ϕ Fourier modes.

The solvability condition of Eq. (A20) reads

$$\frac{q_r}{m_r}e_z^{(2)}\partial_{v_\parallel}F_r^{(0)}-\frac{q_r}{2cm_r}\mathcal{D}^+\mathcal{D}F_r^{(0)}b_x^{(1)}\frac{b_y^{(1)}}{B_0}$$
$$-\frac{q_r}{cm_r}\mathcal{D}^+\langle\sin\phi\partial_\phi f_r^{(1)}\rangle b_y^{(1)}-v_\perp\sin\alpha\partial_\xi\langle\sin\phi\partial_\phi f_r^{(1)}\rangle$$
$$+(v_\parallel\cos\alpha-V_0)\partial_\xi\overline{f_r^{(1)}}=0, \quad (A25)$$

where $\mathcal{D}^\pm=(\Lambda_0-v_\parallel)(\partial_{v_\perp}\pm v_\perp^{-1})+v_\perp\partial_{v_\parallel}$ and

$$\langle\sin\phi\partial_\phi f_r^{(1)}\rangle=-\frac{1}{2}\mathcal{D}\mathcal{F}_r^{(0)}\frac{b_x^{(1)}}{B_0}+\frac{\cos\alpha}{2\Omega_r}(v_\parallel$$
$$-\Lambda_0)\mathcal{D}\mathcal{F}_r^{(0)}\partial_\xi\frac{b_y^{(1)}}{B_0}. \quad (A26)$$

Assuming that the perturbations of the distribution function vanish at large ξ, we obtain

$$(v_\parallel-\Lambda_0)\overline{f_r^{(1)}}=(v_\parallel-\Lambda_0)R_r+S_r\partial_{v_\parallel}F_r^{(0)} \quad (A27)$$

with

$$R_r=\frac{1}{2}\mathcal{D}^+\mathcal{D}\mathcal{F}_r^{(0)}\frac{b_y^{(1)2}}{2B_0^2}-\frac{v_\perp}{2}\partial_{v_\perp}\mathcal{F}_r^{(0)}A$$
$$+\frac{\sin\alpha}{2\Omega_r}v_\perp\mathcal{D}\mathcal{F}_r^{(0)}\partial_\xi\frac{b_y^{(1)}}{B_0}, \quad (A28)$$

$$S_r=\frac{q_r}{m_r}\varphi+\frac{1}{2}v_\perp^2A, \quad (A29)$$

where we have defined $A=b_z^{(1)}/B_0+b_y^{(1)2}/2B_0^2$ and $e_z^{(2)}=-\partial_z\varphi=\cos\alpha\partial_\xi\varphi$. As done in Ref. 16, the potential φ can be determined in terms of the magnetic perturbations A, using Eq. (A4) that, to leading order, gives

$$\sin\alpha\frac{\Lambda_0}{c}\partial_\xi b_y^{(1)}=4\pi\sum_r q_rn_r\int\overline{f_r^{(1)}}d^3v. \quad (A30)$$

We do not use this approach here, but rather eliminate the potential using the expression of the density perturbations.[12]

Furthermore, one has[16]

$$\int\overline{f_r^{(1)}}dv_\parallel=\int R_rdv_\parallel+\mathcal{G}_rS_r, \quad (A31)$$

where we have defined the operator

$$\mathcal{G}_r=\mathrm{P}\int\frac{\partial_{v_\parallel}F_r^{(0)}}{v_\parallel-\lambda}dv_\parallel+\pi(\partial_{v_\parallel}F_r^{(0)})\Big|_{v_\parallel=\lambda}\mathcal{H}_\xi, \quad (A32)$$

$\mathcal{H}_\xi$ being the Hilbert transform with respect to the ξ variable.

The z component of Eq. (A3) (together the previously obtained condition $e_z^{(1)}=0$), leads to the relation

$$\sin\alpha\partial_\xi b_y^{(1)}=\frac{4\pi}{c}\sum_r q_rn_r\int v_\parallel\overline{f_r^{(1)}}d^3v. \quad (A33)$$

The x component of Eq. (A3) taken to leading order gives

$$-\cos\alpha\partial_\xi b_y^{(1)}=\frac{4\pi}{c}\sum_r q_rn_r\int v_\perp\cos\phi\widetilde{f}_r^{(1)}d^3v$$
$$=-\frac{4\pi}{c}\sum_r q_rn_r\int v_\perp\sin\phi\partial_\phi\widetilde{f}_r^{(1)}d^3v, \quad (A34)$$

which, when using (A19), provides the dispersion relation for oblique Alfvén waves, in the form

$$\Lambda_0^2=v_A^2+\frac{p_\perp^{(0)}}{\rho^{(0)}}-\frac{p_\parallel^{(0)}}{\rho^{(0)}}. \quad (A35)$$

It involves the parallel and transverse pressures $p_\parallel^{(0)}=\Sigma_r\,p_{\parallel r}^{(0)}$ and $p_\perp^{(0)}=\Sigma_r\,p_{\perp r}^{(0)}$, together with the corresponding density $\rho^{(0)}=\Sigma_r\,\rho_r^{(0)}$. Here, $p_{\parallel r}^{(0)}=m_rn_r\int v_\parallel^2F_r^{(0)}d^3v$, $p_{\perp r}^{(0)}=m_rn_r\int(v_\perp^2/2)F_r^{(0)}d^3v$, and $\rho_r^{(0)}=m_rn_r\int F_r^{(0)}d^3v$ denote the contributions of the various species to the above quantities. Furthermore, $v_A^2=B_0^2/4\pi\rho^{(0)}$ is the Alfvén velocity.

Finally, the y component of Eq. (A3) gives

$$\partial_\xi(\cos\alpha b_x^{(1)}-\sin\alpha b_z^{(1)})=\frac{4\pi}{c}\sum_r q_rn_r\int v_\perp\sin\phi\widetilde{f}_r^{(2)}d^3v, \quad (A36)$$

which also rewrites

$$-\frac{1}{\sin\alpha}\partial_\xi\frac{b_z^{(1)}}{B_0}=\frac{4\pi}{cB_0}\sum_r q_rn_r\int v_\perp\cos\phi\partial_\phi\widetilde{f}_r^{(2)}d^3v. \quad (A37)$$

It follows that:

$$-\frac{1}{\sin\alpha}\partial_\xi\frac{b_z^{(1)}}{B_0}=-\frac{4\pi}{cB_0}\sum_r q_r n_r\int(\Lambda_0-v_\parallel)\overline{f_r^{(1)}}d^3v\frac{b_y^{(1)}}{B_0}+\frac{4\pi}{B_0^2}\sin\alpha\sum_r m_r n_r\int\frac{v_\perp^2}{2}\partial_\xi\times\left(\overline{f_r^{(1)}}-\frac{1}{2}\langle\sin 2\phi\partial_\phi\widetilde{f}_r^{(1)}\rangle\right)d^3v-\frac{4\pi}{B_0}\cos\alpha\sum_r m_r n_r\int v_\perp(v_\parallel-\Lambda_0)\partial_\xi\langle\sin\phi\partial_\phi\widetilde{f}_r^{(1)}\rangle d^3v. \quad (A38)$$

We are then led to compute

$$\langle\sin 2\phi\partial_\phi\widetilde{f}_r^{(1)}\rangle=\frac{1}{2}\mathcal{D}^-\mathcal{D}F_r^{(0)}\frac{b_y^{(1)2}}{2B_0^2}+\frac{\sin\alpha}{4\Omega_r}v_\perp\mathcal{D}F_r^{(0)}\partial_\xi\frac{b_y^{(1)}}{B_0}. \quad (A39)$$

Using Eqs. (A30) and (A33) (with the condition $\Lambda_0\ll c$) together with the relation

$$\sum_r m_r n_r\int\frac{v_\perp^2}{2}f_r^{(1)}d^3v=p_\perp^{(1)}+\frac{\cos^2\alpha}{\sin\alpha}\partial_\xi\frac{b_y^{(1)}}{B_0}, \quad (A40)$$

where, as previously, we neglect the mass m_e of the electrons compared to that m_p of the protons and defined $v_{\Delta r}^2=(p_{\perp r}^{(0)}-p_{\parallel r}^{(0)})/\rho^{(0)}$.

APPENDIX B: KINETIC FORM OF HYDRODYNAMIC QUANTITIES

1. Density fluctuations

The density fluctuations of the particles of r species, defined to leading order as $\epsilon\rho_r^{(1)}$ with

$$\rho_r^{(1)}=m_r n_r\int\overline{f_r^{(1)}}d^3v, \quad (B1)$$

are given by

$$\rho_r^{(1)}=\mathcal{P}_r\varphi+(\rho_r^{(0)}+\mathcal{O}_r)A-\frac{\sin\alpha}{\Omega_r}\Lambda_0\rho_r^{(0)}\partial_\xi\left(\frac{b_y^{(1)}}{B_0}\right), \quad (B2)$$

where

$$\mathcal{P}_r=2\pi q_r n_r\int_0^\infty\mathcal{G}_r d\left(\frac{v_\perp^2}{2}\right), \quad (B3)$$

$$\mathcal{O}_r=2\pi\sum_r m_r n_r\int_0^\infty\frac{v_\perp^2}{2}\mathcal{G}_r d\left(\frac{v_\perp^2}{2}\right). \quad (B4)$$

The total density fluctuations are then given by $\rho^{(1)}=\sum_r\rho_r^{(1)}$.

2. Hydrodynamic velocities

The hydrodynamic velocity transverse to the local magnetic field is given by

$$U_\perp=\frac{\sum_r m_r n_r\int V_\perp f_r d^3v}{\sum_r m_r n_r\int f_r d^3v}, \quad (B5)$$

where $V_\perp=v-(v\cdot\hat{b})\hat{b}$ with $\hat{b}=b/|b|$. One easily checks that $V_\perp\equiv(V_{\perp x},V_{\perp y},V_{\perp z})$ with

$$V_{\perp x}=v_\perp\cos\phi-\epsilon v_\parallel\frac{b_x^{(1)}}{B_0}, \quad (B6)$$

$$V_{\perp y}=v_\perp\sin\phi-\epsilon^{1/2}v_\parallel\frac{b_y^{(1)}}{B_0}-\epsilon v_\perp\sin\phi\frac{b_y^{(1)2}}{B_0^2}, \quad (B7)$$

$$V_{\perp z}=-\epsilon^{1/2}v_\perp\sin\phi\frac{b_y^{(1)}}{B_0}+\epsilon\left(2v_\parallel\frac{b_y^{(1)2}}{2B_0^2}-v_\perp\cos\phi\frac{b_x^{(1)}}{B_0}\right), \quad (B8)$$

and thus

$$U_\perp\equiv(U_{\perp x},U_{\perp y},U_{\perp z})=\left(-\epsilon\Lambda_0\frac{b_x^{(1)}}{B_0}-\epsilon\frac{\cos\alpha}{\Omega_p}(v_A^2+v_{\Delta e}^2)\partial_\xi\left(\frac{b_y^{(1)}}{B_0}\right),-\epsilon^{1/2}\Lambda_0\frac{b_y^{(1)}}{B_0},\epsilon\Lambda_0\frac{b_y^{(1)2}}{B_0^2}\right), \quad (B9)$$

where terms in m_e/m_p have been neglected.

The hydrodynamic velocity along the local magnetic field is

$$U_\parallel=\frac{\sum_r m_r n_r\int V_\parallel f_r d^3v}{\sum_r m_r n_r\int f_r d^3v} \quad (B10)$$

with $V_\parallel=(v\cdot\hat{b})\hat{b}$. One gets $U_\parallel=\epsilon U_\parallel^{(1)}+\cdots$ with

$$U_\parallel^{(1)}=-\Lambda_0\frac{b_y^{(1)2}}{2B_0^2}+\frac{\Lambda_0}{\rho^{(0)}}\mathcal{P}\varphi+\frac{\Lambda_0}{\rho^{(0)}}\mathcal{O}A-\frac{\sin\alpha}{\Omega_p}v_{\Delta p}^2\partial_\xi\frac{b_y^{(1)}}{B_0} \quad (B11)$$

that also rewrites

$$U_\parallel^{(1)}=-\Lambda_0\frac{b_y^{(1)2}}{2B_0^2}+\Lambda_0\frac{\rho^{(1)}}{\rho^{(0)}}-\Lambda_0 A+\frac{\sin\alpha}{\Omega_p}(\Lambda_0^2-v_{\Delta p}^2)\partial_\xi\frac{b_y^{(1)}}{B_0}. \quad (B12)$$

By projecting $U_\perp+U_\parallel\hat{b}$ on the three axes, we recover the hydrodynamic velocity components

$$u_x\approx-\epsilon\left(\Lambda_0\frac{b_x^{(1)}}{B_0}+(v_A^2+v_{\Delta e}^2)\frac{\cos\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0}\right), \quad (B13)$$

$$u_y\approx-\epsilon^{1/2}\Lambda_0\frac{b_y^{(1)}}{B_0}, \quad (B14)$$

$$u_z\approx\epsilon\left[\Lambda_0\left(\frac{\rho^{(1)}}{\rho^{(0)}}-\frac{b_z^{(1)}}{B_0}\right)+(v_A^2+v_{\Delta e}^2)\frac{\sin\alpha}{\Omega_p}\partial_\xi\frac{b_y^{(1)}}{B_0}\right]. \quad (B15)$$

3. Gyrotropic pressures

In the framework of a monofluid theory, the transverse and parallel components of the gyrotropic pressures are defined as

$$p_{\perp r} = m_r n_r \int \frac{1}{2}(V_\perp - U_\perp)^2 f_r d^3v, \tag{B16}$$

$$p_{\parallel r} = m_r n_r \int (V_\parallel - U_\parallel)^2 f_r d^3v. \tag{B17}$$

Defining the operators

$$\mathcal{M} = \sum_r \mathcal{M}_r = 2\pi \sum_r q_r n_r \int_0^\infty d\left(\frac{v_\perp^2}{2}\right)\frac{v_\perp^2}{2}\mathcal{G}_r, \tag{B18}$$

$$\mathcal{N} = \sum_r \mathcal{N}_r = 2\pi \sum_r m_r n_r \int_0^\infty d\left(\frac{v_\perp^2}{2}\right)\frac{v_\perp^4}{4}\mathcal{G}_r, \tag{B19}$$

the leading pressure perturbations (or order ϵ) are given by

$$p_{\perp r}^{(1)} = m_r n_r \int \frac{v_\perp^2}{2}\overline{f_r^{(1)}} d^3v - \left(\Lambda_0^2 \frac{\rho_r^{(0)}}{\rho^{(0)}} - v_{\Delta r}^2\right)\rho^{(0)}\frac{b_y^{(1)2}}{2B_0^2}$$
$$= \mathcal{M}_r\varphi + (2p_{\perp r}^{(0)} + \mathcal{N}_r)A - 2\frac{\sin\alpha}{\Omega_r}\Lambda_0 p_{\perp r}^{(0)}\partial_\xi \frac{b_y^{(1)}}{B_0} \tag{B20}$$

and

$$p_{\parallel r}^{(1)} = m_r n_r \int \frac{v_\parallel^2}{2}\overline{f_r^{(1)}} d^3v - v_{\Delta r}^2 \rho^{(0)}\frac{b_y^{(1)2}}{2B_0^2}$$
$$= (-q_r n_r + \Lambda_0^2 \mathcal{P}_r)\varphi + (-\rho^{(0)} v_{\Delta r}^2 + \Lambda_0^2 \mathcal{O}_r)A - \frac{\sin\alpha}{\Omega_r}\Lambda_0 p_{\parallel r}^{(0)}\partial_\xi \frac{b_y^{(1)}}{B_0}. \tag{B21}$$

4. Heat fluxes

a. Gyrotropic heat fluxes

The gyrotropic components of the heat flux tensor

$$\mathbf{q}_r = \sum_r m_r n_r \int (v-U)\otimes(v-U)\otimes(v-U) f_r d^3v, \tag{B22}$$

(U denoting the hydrodynamic velocity) read

$$q_{\perp r} = m_r n_r \int \frac{1}{2}(V_\perp - U_\perp)^2 (V_\parallel - U_\parallel) f_r d^3v, \tag{B23}$$

$$q_{\parallel r} = m_r n_r \int (V_\parallel - U_\parallel)^3 f_r d^3v. \tag{B24}$$

To leading order, one has $q_{\perp r} = \epsilon q_{\perp r}^{(1)} + \cdots$ and $q_{\parallel r} = \epsilon q_{\parallel r}^{(1)} + \cdots$ where

$$q_{\perp r}^{(1)} = -p_{\perp r}^{(0)}\left(U_\parallel^{(1)} + \Lambda_0 \frac{b_y^{(1)2}}{2B_0^2}\right) + \Lambda_0 \mathcal{M}_r \varphi + \Lambda_0 \mathcal{N}_r A$$
$$+ \frac{\sin\alpha}{\Omega_r} m_r n_r \int \left(v_\parallel^2 v_\perp^2 - \frac{1}{4}v_\perp^4\right) F_r^{(0)} d^3v\, \partial_\xi \frac{b_y^{(1)}}{B_0}, \tag{B25}$$

$$q_{\parallel r}^{(1)} = -3p_{\parallel r}^{(0)}\left(U_\parallel^{(1)} + \Lambda_0 \frac{b_y^{(1)2}}{2B_0^2}\right) + \Lambda_0^3 \rho_r^{(1)} - \Lambda_0(\Lambda_0^2 \rho_r^{(0)}$$
$$+ p_{\perp r}^{(0)})A - q_r n_r \Lambda_0 \varphi + \frac{\sin\alpha}{\Omega_r}\Bigg[\Lambda_0^4 \rho_r^{(0)} + m_r n_r$$
$$\times \int \left(v_\parallel^4 - \frac{3}{2}v_\perp^2 v_\parallel^2\right) F_r^{(0)} d^3v\Bigg]\partial_\xi \frac{b_y^{(1)}}{B_0}, \tag{B26}$$

with $U_\parallel^{(1)} + \Lambda_0 b_y^{(1)2}/2B_0^2$ given by Eq. (B11).

b. Heat flux contributions to the gyrotropic pressures

The longitudinal and transverse pressures (relatively to the local magnetic field) involve tr $\boldsymbol{\nabla}\cdot\mathbf{q}_r$ and $\hat{b}\cdot\boldsymbol{\nabla}\cdot\mathbf{q}_r\cdot\hat{b}$. The heat flux components being also of order ϵ, the distortion of the magnetic field lines can be neglected to leading order. We are thus led to write

$$\text{tr } \boldsymbol{\nabla}\cdot\mathbf{q}_r \approx \partial_x(q_{r111} + q_{r221} + q_{r331}) + \partial_z(q_{\perp r} + q_{\parallel r}) \tag{B27}$$

and

$$\hat{b}\cdot(\boldsymbol{\nabla}\cdot\mathbf{q}_r)\cdot\hat{b} \approx \partial_x q_{r331} + \partial_z q_{\parallel r}, \tag{B28}$$

where the q_{rijk}'s hold for the components in the local frame of the heat flux associated with the particles of species r. One has

$$q_{r331} \equiv \int (V_\parallel - U_\parallel)^2 (V_{\perp x} - U_{\perp x}) f_r d^3v = \epsilon q_{r331}^{(1)} + \cdots \tag{B29}$$

with

$$q_{r331}^{(1)} = \Bigg\{\frac{\cos\alpha}{\Omega_p} p_{\parallel r}^{(0)}\left[v_A^2 + v_{\Delta e}^2 - \frac{\Omega_p}{\Omega_r}(v_A^2 + v_{\Delta p}^2 + v_{\Delta e}^2)\right]$$
$$- \frac{\cos\alpha}{\Omega_r}\int \left(v_\parallel^4 - \frac{3}{2}v_\perp^2 v_\parallel^2\right) F_r^{(0)} d^3v\Bigg\}\partial_\xi \frac{b_y^{(0)}}{B_0}. \tag{B30}$$

Similarly, to leading order

$$q_{r111} + q_{r221} = \int (V_\perp - U_\perp)^2 (V_{\perp x} - U_{\perp x}) f_r d^3v$$
$$= \epsilon (q_{111,r}^{(1)} + q_{221,r}^{(1)}) + \cdots \tag{B31}$$

with

$$q_{r111}^{(1)} + q_{r221}^{(1)} = \Bigg\{-\frac{\cos\alpha}{\Omega_r}\Bigg[4p_{\perp r}^{(0)}\Lambda_0^2 + 2m_r n_r$$
$$\times \int \left(v_\parallel^2 v_\perp^2 - \frac{1}{4}v_\perp^2\right) F_r^{(0)} d^3v\Bigg]$$
$$+ 4\frac{\cos\alpha}{\Omega_p} p_{\perp r}^{(0)}(v_A^2 + v_{\Delta e}^2)\Bigg\}\partial_\xi \frac{b_y^{(0)}}{B_0}. \tag{B32}$$

c. Heat flux contribution to the nongyrotropic pressures

Neglecting the magnetic field line distortions that are irrelevant at the considered order, we write the nongyrotropic contribution to $\boldsymbol{\nabla}\cdot\mathbf{q}_p$ in the form

$$\overline{\overline{\boldsymbol{\nabla}\cdot\mathbf{q}_p}} = \partial_k\begin{pmatrix} \frac{1}{2}(q_{p11k}-q_{p22k}) & q_{p12k} & q_{p13k} \\ q_{p21k} & -\frac{1}{2}(q_{p11k}-q_{p22k}) & q_{p23k} \\ q_{p31k} & q_{p32k} & 0 \end{pmatrix} \tag{B33}$$

where we concentrate on the proton contribution. To leading order $q_{pijk}=\int(v_i-u_i)(v_j-u_j)(v_k-u_k)f_p d^3v \approx \epsilon q^{(1)}_{pijk}$. Since we assume no dependency in the y variable, we are led to compute

$$q^{(1)}_{p111} = -3p^{(0)}_{\perp p}u^{(1)}_x + m_p n_p \int v_\perp^3 \cos^3\phi f^{(1)}_p d^3v, \tag{B34}$$

$$q^{(1)}_{p221} = -p^{(0)}_{\perp p}u^{(1)}_x + m_p n_p \int v_\perp^3 \sin^2\phi \cos\phi f^{(1)}_p d^3v, \tag{B35}$$

$$q^{(1)}_{p112} = m_p n_p \int v_\perp^3 \cos^2\phi \sin\phi f^{(1)}_p d^3v, \tag{B36}$$

$$q^{(1)}_{p113} = -p^{(0)}_{\perp p}u^{(1)}_z + m_p n_p \int v_\perp^2 v_\parallel \cos^2\phi f^{(1)}_p d^3v, \tag{B37}$$

$$q^{(1)}_{p223} = -p^{(0)}_{\perp p}u^{(1)}_z + 2(p^{(0)}_{\perp p}-p^{(0)}_{\parallel p})u_y\frac{b^{(1)}_y}{B_0} + m_p n_p \int v_\perp^2 v_\parallel \sin^2\phi f^{(1)}_p d^3v, \tag{B38}$$

$$q^{(1)}_{p123} = m_p n_p \int v_\perp^2 v_\parallel \sin\phi \cos\phi f^{(1)}_p d^3v, \tag{B39}$$

$$q^{(1)}_{p133} = -p^{(0)}_{\parallel p}u^{(1)}_x + m_p n_p \int v_\perp v_\parallel^2 \cos\phi f^{(1)}_p d^3v, \tag{B40}$$

$$q^{(1)}_{p233} = m_p n_p \int v_\perp v_\parallel^2 \sin\phi f^{(1)}_p d^3v. \tag{B41}$$

Since $f^{(1)}_p$ only projects on 1, $\cos\phi$, and $\cos 2\phi$, one has

$$q^{(1)}_{p112} = q^{(1)}_{p123} = q^{(1)}_{p233} = 0, \tag{B42}$$

and the only integrals to be computed read

$$m_p n_p \int v_\perp^3 \sin\phi\, \partial_\phi f^{(1)}_p d^3v = 4\Lambda_0 p^{(0)}_{\perp p}\frac{b^{(1)}_x}{B_0} + \left[4\Lambda_0^2 p^{(0)}_{\perp p} + 4m_p n_p \times \int\left(v_\parallel^2 v_\perp^2 - \frac{v_\perp^4}{4}\right)F^{(0)}_p d^3v\right]\frac{\cos\alpha}{\Omega_p}\partial_\xi\frac{b^{(1)}_y}{B_0}, \tag{B43}$$

$$m_p n_p \int v_\perp v_\parallel^2 \sin\phi\, \partial_\phi f^{(1)}_p d^3v = \Lambda_0 p^{(0)}_{\parallel p}\frac{b^{(1)}_x}{B_0} + \left(\Lambda_0^2 p^{(0)}_{\parallel p} + m_p n_p \int\left(v_\parallel^4 - \frac{3}{2}v_\perp^2 v_\parallel^2\right)F^{(0)}_p d^3v\right) \times\frac{\cos\alpha}{\Omega_p}\partial_\xi\frac{b^{(1)}_y}{B_0}, \tag{B44}$$

$$m_p n_p \int v_\perp^2 v_\parallel \sin 2\phi\, \partial_\phi f^{(1)}_p d^3v = -8\Lambda_0(p^{(0)}_{\perp p}-p^{(0)}_{\parallel p})\frac{b^{(1)2}_y}{2B_0^2} + \int\left(v_\parallel^2 v_\perp^2 - \frac{v_\perp^4}{4}\right) \times F^{(0)}_p d^3v\frac{\sin\alpha}{\Omega_p}\partial_\xi\frac{b^{(1)}_y}{B_0}. \tag{B45}$$

The fourth-order velocity moments are explicited in Appendix C where bi-Maxwellian distribution functions are assumed for the equilibrium state.

APPENDIX C: EQUILIBRIUM BI-MAXELLIAN DISTRIBUTION

It is possible to simplify the above general expressions for the hydrodynamic moments by assuming that the plasma contains electrons and only one species of ions (with $Z=1$), with bi-Maxwellian equilibrium distribution functions

$$F^{(0)}_r = \frac{1}{(2\pi)^{3/2}}\frac{m_r^{3/2}}{T^{(0)}_{\perp r}T^{(0)1/2}_{\parallel r}}\exp\left\{-\left(\frac{m_r}{2T^{(0)}_{\parallel r}}v_\parallel^2 + \frac{m_r}{2T^{(0)}_{\perp r}}v_\perp^2\right)\right\}. \tag{C1}$$

Using the quasineutrality condition that prescribes $n_r=n^{(0)}$ and $\rho^{(1)}_r=m_r n^{(1)}$, one obtains

$$\mathcal{M}_r = -n^{(0)}q_r\frac{T^{(0)}_{\perp r}}{T^{(0)}_{\parallel r}}\mathcal{W}_r, \quad \mathcal{N}_r = -2n^{(0)}\frac{T^{(0)2}_{\perp r}}{T^{(0)}_{\parallel r}}\mathcal{W}_r, \tag{C2}$$

$$\mathcal{O}_r = -n^{(0)}m_r\frac{T^{(0)}_{\perp r}}{T^{(0)}_{\parallel r}}\mathcal{W}_r, \quad \mathcal{P}_r = -n^{(0)}m_r q_r\frac{1}{T^{(0)}_{\parallel r}}\mathcal{W}_r, \tag{C3}$$

where, normalizing the propagation velocity of the wave by the thermal velocity $v_{th,r}=\sqrt{T^{(0)}_{\parallel r}/m_r}$ in the form $c_r=\Lambda_0/v_{th,r}$, one writes

$$\mathcal{W}_r \equiv \mathcal{W}(c_r) = \frac{1}{\sqrt{2\pi}}\mathrm{P}\int\frac{\zeta e^{-\zeta^2/2}}{\zeta-c_r}d\zeta + \sqrt{\frac{\pi}{2}}c_r e^{-c_r^2/2}\mathcal{H}_\xi, \tag{C4}$$

or[43]

$$\mathcal{W}(c_r) = 1 - c_r e^{-\frac{c_r^2}{2}}\int_0^{c_r} e^{\frac{\zeta^2}{2}} d\zeta + \sqrt{\frac{\pi}{2}} c_r e^{-c_r^2/2}\mathcal{H}_\xi. \qquad \text{(C5)}$$

This function is related to the plasma response function $\mathcal{R}$ used by SHD by $\mathcal{W}(X)=\mathcal{R}(X/\sqrt{2})$.

This leads to express

$$\frac{p_{\parallel r}^{(1)}}{p_{\parallel r}^{(0)}} = \left(1 - \frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}} - \frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}} c_r^2 \mathcal{W}_r\right)A + (c_r^2 + \mathcal{W}_r^{-1})\left[\frac{n^{(1)}}{n^{(0)}} - \left(1 - \frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}}\mathcal{W}_r\right)A\right] + (c_r^2 - 1 + \mathcal{W}_r^{-1})\frac{\sin\alpha}{\Omega_r}\Lambda_0 \partial_\xi \frac{b_y^{(1)}}{B_0} \qquad \text{(C6)}$$

and

$$\frac{T_{\parallel r}^{(1)}}{T_{\parallel r}^{(0)}} \equiv \frac{p_{\parallel r}^{(1)}}{p_{r}^{(0)}} - \frac{n^{(1)}}{n^{(0)}} = (c_r^2 - 1 + \mathcal{W}_r^{-1}) \times \left(\frac{n^{(1)}}{n^{(0)}} - A + \frac{\sin\alpha}{\Omega_r}\Lambda_0\partial_\xi \frac{b_y^{(1)}}{B_0}\right). \qquad \text{(C7)}$$

Similarly,

$$\frac{T_{\perp r}^{(1)}}{T_{\perp r}^{(0)}} \equiv \frac{p_{\perp r}^{(1)}}{p_{\perp r}^{(0)}} - \frac{n^{(1)}}{n^{(0)}} = \left(1 - \frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}}\mathcal{W}_r\right)A - 3\frac{\sin\alpha}{\Omega_r} c_r v_{thr} \partial_\xi \frac{b_y^{(1)}}{B_0}. \qquad \text{(C8)}$$

When considering the heat flux components, we have to evaluate the integrals

$$m_r n_r \int \left(v_\parallel^4 - \frac{3}{2} v_\parallel^2 v_\perp^2\right) F_r^{(0)} d^3 v = -3 v_{\Delta r}^2 p_r^{(0)} \qquad \text{(C9)}$$

and

$$m_r n_r \int \left(v_\parallel^2 v_\perp^2 - \frac{1}{4} v_\perp^4\right) F_r^{(0)} d^3 v = -2 v_{\Delta r}^2 p_r^{(0)}. \qquad \text{(C10)}$$

We get

$$\frac{q_{\parallel r}^{(1)}}{v_{th,r} p_{\parallel r}^{(0)}} = c_r \frac{c_r^2 - 3 + W_r^{-1}}{c_r^2 - 1 + W_r^{-1}} \frac{T_{\parallel r}^{(1)}}{T_{\parallel r}^{(0)}} + 3\left[\left(c_r^2 - \frac{v_{\Delta r}^2}{v_{thr}^2}\right)\left(\frac{\Omega_p}{\Omega_r} - 1\right) + \frac{v_{\Delta p}^2 - v_{\Delta r}^2}{v_{th,r}^2}\right]\frac{\sin\alpha}{\Omega_p} v_{th,r}\partial_\xi \frac{b_y^{(1)}}{B_0} \qquad \text{(C11)}$$

and

$$\frac{q_{\perp r}^{(1)}}{v_{th,r} p_{\perp r}^{(0)}} = -\frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}} c_r W_r A + \left[-c_r^2\left(\frac{\Omega_p}{\Omega_r} + 1\right) + \frac{v_{\Delta p}^2}{v_{th,r}^2} - 2\frac{v_{\Delta r}^2}{v_{th,r}^2}\frac{\Omega_p}{\Omega_r}\right]\frac{\sin\alpha}{\Omega_p} v_{th,r}\partial_\xi \frac{b_y^{(1)}}{B_0} \qquad \text{(C12)}$$

that also rewrites

$$\frac{q_{\perp r}^{(1)}}{v_{th,r} p_{\perp r}^{(0)}} = -\frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}} \frac{c_r W_r}{1 - \frac{T_{\perp r}^{(0)}}{T_{\parallel r}^{(0)}} W_r}\left[\frac{T_{\perp r}^{(1)}}{T_{\perp r}^{(0)}} + 3 c_r \frac{\sin\alpha}{\Omega_r} v_{th,r}\partial_\xi \frac{b_y^{(1)}}{B_0}\right] + \left[-c_r^2\left(\frac{\Omega_p}{\Omega_r} + 1\right) + \frac{v_{\Delta p}^2}{v_{th,r}^2} - 2\frac{v_{\Delta r}^2}{v_{th,r}^2}\frac{\Omega_p}{\Omega_r}\right]\frac{\sin\alpha}{\Omega_p} v_{th,r}\partial_\xi \frac{b_y^{(1)}}{B_0}. \qquad \text{(C13)}$$

Note that for $\alpha=0$, the parallel and transverse energy fluxes computed in Refs. 16 and 12 in the case of parallel Alfvén waves are recovered.

Furthermore, the nonzero coefficients entering the nongyrotropic proton heat flux components considered in Appendix B, become

$$q_{p111}^{(1)} - q_{p221}^{(1)} = 0, \qquad \text{(C14)}$$

$$q_{p113}^{(1)} - q_{p223}^{(1)} = p_{\perp p}^{(0)} v_{\Delta p}^2 \frac{\sin\alpha}{\Omega_p}\partial_\xi \frac{b_y^{(1)}}{B_0}, \qquad \text{(C15)}$$

$$q_{p133}^{(1)} = 2 p_{\parallel p}^{(0)} v_{\Delta p}^2 \frac{\cos\alpha}{\Omega_p}\partial_\xi \frac{b_y^{(1)}}{B_0}, \qquad \text{(C16)}$$

together with

$$\begin{aligned} q_{p113}^{(1)} &= -p_{\perp p}^{(0)} u_z^{(1)} + \Lambda_0 p_{\perp p}^{(1)} - 2\Lambda_0 p_{\perp p}^{(0)} A + \Lambda_0 p_{\perp p}^{(0)} \frac{b_y^{(1)2}}{2B_0^2} + \left(2\Lambda_0^2 - \frac{3}{2} v_{\Delta p}^2\right) p_{\perp p}^{(0)} \frac{\sin\alpha}{\Omega_p}\partial_\xi \frac{b_y^{(1)}}{B_0} \\ &= \Lambda_0 p_{\perp p}^{(0)}\left(\frac{p_{\perp p}^{(1)}}{p_{\perp p}^{(0)}} - \frac{\rho_p^{(1)}}{\rho_p^{(0)}} - A\right) + \left(\Lambda_0^2 - \frac{v_{\Delta p}^2}{2}\right) p_{\perp p}^{(0)} \frac{\sin\alpha}{\Omega_p}\partial_\xi \frac{b_y^{(1)}}{B_0} \\ &= q_{\perp p}^{(1)} + p_{\perp p}^{(0)} v_{\Delta p}^2 \frac{\sin\alpha}{2\Omega_p}\partial_\xi \frac{b_y^{(1)}}{B_0}, \end{aligned} \qquad \text{(C17)}$$

where we have used Eqs. (B24), (B11), and (B19).

Equation (B32) then reads

$$\overline{\overline{\boldsymbol{\nabla}\cdot\mathbf{q}_p}} = \begin{pmatrix} \lambda_p & 0 & \mu_p \\ 0 & -\lambda_p & 0 \\ \mu_p & 0 & 0 \end{pmatrix} \qquad \text{(C18)}$$

with

$$\lambda_p = p_{\perp p}^{(0)} \frac{v_{\Delta p}^2}{2\Omega_p}\partial_{xz}\frac{b_y^{(1)}}{B_0}, \qquad \text{(C19)}$$

$$\mu_p = \partial_x q_{\perp p} + p_{\perp p}^{(0)} \frac{v_{\Delta p}^2}{2\Omega_p}\partial_{xx}\frac{b_y^{(1)}}{B_0} + 2 p_{\parallel p}^{(0)} \frac{v_{\Delta p}^2}{\Omega_p}\partial_{zz}\frac{b_y^{(1)}}{B_0}. \qquad \text{(C20)}$$

[1] D. H. E. Dubin, J. A. Krommes, C. Oberman, and W. W. Lee, Phys. Fluids **26**, 3524 (1983).
[2] W. W. Lee, Phys. Fluids **26**, 556 (1983).
[3] G. F. Chew, M. L. Goldberger, and F. E. Low, Proc. R. Soc. London, Ser. A **236**, 112 (1956).

[4]G. Belmont and L. Rezeau, Ann. Geophys., Ser. A **5**, A59 (1987).
[5]R. M. Kulsrud, in *Handbook of Plasma Physics*, edited by M. N. Rosenbluth and R. Z. Sagdeev (New Holland, New York, 1983), Vol. 1; *Basic Plasma Physics*, edited by A. A. Galeev and R. N. Sudan (North Holland, New York, 1983), pp. 115–145.
[6]G. Belmont and C. Mazelle, J. Geophys. Res., [Space Phys.], **97**, No. A6, 8327 (1992).
[7]K. M. Ferrière and N. André, J. Geophys. Res., [Space Phys.], **107**, No. A11, 1349 (2002).
[8]A. Brizard, Phys. Fluids B **4**, 1213 (1992).
[9]W. Dorland and G. W. Hammett, Phys. Fluids B **5**, 812 (1993).
[10]C. Z. Cheng and J. R. Johnson, J. Geophys. Res., [Space Phys.] **104** No. A1, 413 (1998JA900065) (1999).
[11]P. B. Snyder, G. W. Hammett, and W. Dorland, Phys. Plasmas **4**, 3974 (1997).
[12]T. Passot and P. L. Sulem, Phys. Plasmas **10**, 3906 (2003).
[13]A. Rogister, Phys. Fluids **12**, 2733 (1971).
[14]E. Mjølhus and J. Wyller, J. Plasma Phys. **40**, 299 (1988).
[15]M. V Medvedev and P. H. Diamond, Phys. Plasmas **3**, 863 (1996).
[16]T. Passot and P. L. Sulem, Phys. Plasmas **10**, 3887 (2003).
[17]T. Passot and P. L. Sulem, Nonlinear Processes Geophys. **11**, 245 (2004).
[18]A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Sitenko, and K. N. Stepanov, *Plasma Electrodynamics* (Pergamon, New York, 1975), Vol 1.
[19]A. Hasegawa and L. Chen, Phys. Fluids **19**, 1924 (1976).
[20]N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill, New York, 1973).
[21]C. T. Hsu, R. D. Hazeltine, and P. J. Morrison, Phys. Fluids **29**, 1480 (1986).
[22]Z. Chang and J. D. Callen, Phys. Fluids B **4**, 1766 (1992).
[23]S. I. Braginskii, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, pp. 205–311.
[24]N. Yajima, Prog. Theor. Phys. **36**, 1 (1966).
[25]M. Khanna and R. Rajaram, J. Plasma Phys. **28**, 459 (1982).
[26]V. A. Marchenko, R. E. Denton, and M. K. Hudson, Phys. Plasmas **3**, 3861 (1996).
[27]K. Mio, T. Ogino, and S. Takeda, J. Phys. Soc. Jpn. **41**, 2114 (1976).
[28]A. Gazol, T. Passot, and P. L. Sulem, J. Plasma Phys. **60**, 95 (1998).
[29]A. B. Mikhailovskii and A. I. Smolyakov, Sov. Phys. JETP **61**, 109 (1985) [Zh. Eksp. Teor. Fiz. **88**, 189 (1985)].
[30]D. Chakraborty and K. P. Das, J. Plasma Phys. **64**, 211 (2000).
[31]T. Passot and P. L. Sulem, Phys. Plasmas **10**, 3914 (2003).
[32]O. Alexandrova, A. Mangeney, M. Maksimovic *et al.*, J. Geophys. Res., [Space Phys.], **109**, A05208 (2004).
[33]W. Park, E. V. Belova, G. Y. Fu, X. Z. Tang, H. R. Strauss, and L. E. Sugiyama, Phys. Plasmas **6**, 1796 (1999).
[34]B. G. Elmegreen and J. Scalo, Ann. Rev. Astron. Astrophys. (in press)
[35]B. T. Tsurutani, B. Dasgupta, J. K. Arballo, G. S. Lakhina, and J. S. Pickett, Nonlinear Processes Geophys. **21**, 21 (2003).
[36]K. Stasiewicz, P. K. Shukla, G. Gustafsson, S. Buchert, B. Lavraud, B. Thidé, and Z. Lkos, Phys. Rev. Lett. **90**, 085002 (2003).
[37]D. Winterhalter, M. Neugebauer, B. E. Goldstein, E. J. Smith, B. T. Tsurutani, S. J. Bame, and A. Balogh, Space Sci. Rev. **72**, 201 (1995).
[38]E. A. Lucek, M. W. Dunlop, A. Balogh, P. Cargill, W. Baumjohann, E. Georgescu, G. Haerendel, and K. H. Fornacon, Geophys. Res. Lett. **26**, 2159 (1999).
[39]I. O. Pogutse, A. I. Smolyakov, and A. Hirose, J. Plasma Phys. **60**, 133 (1998).
[40]F. Verheest, J. Plasma Phys. **47**, 25 (1992).
[41]F. Verheest and B. Buti, J. Plasma Phys. **47**, 15 (1992).
[42]I. Kh. Khabibrakhmanov, V. L. Galinsky, and F. Verheest, Phys. Fluids B **4**, 2538 (1992).
[43]S. Ischimaru, *Basic Principles of Plasma Physics: A Statistical Approach*, Frontiers in Physics (Benjamin, New York, 1973).