Heavy quarks in rotating plasma via holography

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Abstract

We continue to explore the dynamics of a heavy quark in a strongly coupled rotating quark-gluon plasma within the holographic framework. We use the 5d Kerr-AdS black hole with two arbitrary rotating parameters to describe the rotating quark-gluon plasma. We present the relations for the drag force acting on the heavy quark both in the rotating- and static-at-infinity frames for arbitrary choice of the rotational parameters. We also calculate the thermal mass of a static quark in different frames and show that at high-T limit the dependence on rotation is suppressed.

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1. Introduction

Studies of black holes in anti-de Sitter (AdS) spacetimes via holography are important since they help to understand features of dual field theories at strong coupling that are inaccessible by perturbative calculations [1–6]. Particularly, the holographic duality provides a link between properties of asymptotically AdS black holes and observables relevant to physics of the quark-gluon plasma (QGP). The QGP represents a hot strongly coupled fluid in the phase of deconfinement and, moreover, at high temperature, the lattice results show near conformal behavior, so the holographic approach seems to be viable for QGP.

Recently there has been some significant interest in the description and study of a rotating QGP formed in heavy-ion collisions. It is known that in non-central heavy-ion collisions there is a nonzero total angular momentum that is related to colliding nuclei. The major part of this angular momentum is taken away by the spectator nucleons; however some amount of angular momentum remains in the QGP and is conserved in time [7–9]. The measurements of the Λ, $\bar{\Lambda}$ hyperon polarization by STAR collaboration at RHIC predict the angular velocity $\Omega \sim 6 \pm 1$ MeV [10–12]. Note that the value of Ω obtained in hydrodynamic simulations is even greater: $\Omega \sim 20 - 40$ MeV [9]. So it is of interest to trace how the fast rotation leads to different phenomena in QGP.

In the holographic context, the rotating QGP can be described by virtue of a 5d asymptotically AdS black hole with a non-zero angular momentum [13–22]. This class of black holes includes the 5d Kerr-AdS solutions constructed in [23]. It is interesting to note that, comparing to the 4-dimensional Kerr-AdS black holes [24], the five dimensional ones have two rotational parameters a and b, i.e. the angular momentum consists of two parts that are conserved independently. The 5-dimensional Kerr-AdS black holes are dual to the thermal $\mathcal{N} = 4$ SYM at strong coupling on the cylinder $\mathbb{R} \times \mathbb{S}^3$ [23,25]. We note that the Kerr-AdS_5 black hole can be described in rotating- (Boyer-Lindquist coordinates) or static-at-the-boundary (asymptotically AdS coordinates) frames. The disadvantage of BL coordinates comparing to AdS ones is an absence of S^3-symmetry near the boundary of the Kerr-AdS_5 spacetime. However, for small values of rotational parameters the boundary in BL coordinates has a similar form to the boundary of the Kerr-AdS spacetime in the static-at-the-boundary frame. The case where both rotational parameters are non-zero can be associated with a gyroscopic motion in the dual field theory, while a single non-zero rotational parameter corresponds to a rotation around some axis. The

Kerr-AdS black holes possess the Hawking-Page transition, which is associated to a confinement-deconfinement phase transition in the dual theory. The rotation of Kerr-AdS black hole affects the phase transition. Namely, the temperature of the phase transition decreases with the rotation, as shown using the holographic approach in the recent work [13]. The similar influence of the rotation on the Hawking-Page transition was found in [20], where the behavior of the free energy and the entropy was studied. Moreover, in the latter work it was shown that the phase transition takes place only for relatively small parameters and disappears if at least one of them is greater than some admissible value ($a, b \in [0; 0.25]$ for $\ell = 1$). These values of the rotational parameters exactly correspond to the angular velocity obtained in the hydrodynamical simulations [9]. It is worth to be noted that in a number of the effective models of rotating QGP [27–30] it was found that the rotation decreases the deconfinement temperature. At the same time, recent studies [31,32] using lattice QCD calculations have shown that the phase transition temperature T_c increases with Ω.

Heavy quarks represent suitable probes of the strongly coupled QGP. Transport properties of QGP can be understood by calculating observables related to heavy quark dynamics in the medium. When moving through the QGP, an energetic quark looses its energy fast because of strong coupling. Energy losses are observed through jet quenching at experiments on RHIC. For heavy quarks, which have a relative large mass, energy losses can be studied more clearly, since low-energy strong-interaction effects are masked by the quark mass.

Using a perturbative QCD approach, the energy of the heavy quark momentum diffusion coefficient was calculated earlier in [33–38]. Another possibility to study jet quenching at strong coupling is the lattice-QCD calculations. The heavy quark diffusion coefficient and the jet quenching coefficient, related with the averaged transverse momentum broadening squared per unit mean free path [39,40], using the lattice approach were discussed in [41–43].

In the holographic framework the energy loss of the heavy quark can be studied by at least two ways. The first one is based on the relation of a jet-quenching coefficient $\hat{q}$ to light-like Wilson loops and comes down to the calculation of holographic Wilson loops [44].

The second way is to calculate the drag force acting on a heavy quark in the thermal medium. In this case the heavy quark is associated with an endpoint of an open string stretched in the black hole background from the boundary of the spacetime down to the horizon [45,46]. Within the second approach in the ordinary and charged AdS black branes the energy loss was broadly studied in [47–56]. In the framework of the fluid/gravity correspondence the drag force was studied in [57,58].

In works [17,20] the holographic drag force in the rotating QGP was calculated using 4d and 5d Kerr-AdS black holes, respectively. In the paper [20] the drag force was found for the cases of the 5d Kerr-AdS black hole with one non-zero and two equal rotational parameters. It was shown in [20] that if the 5d black hole has just one rotational parameter the relation for the drag force is in agreement with the result of [17]; at the same time it was obtained that the drag force vanishes for the case of two equal rotational parameters.

The purpose of this paper is to present generic relations for the drag force acting on a heavy quark in a rotating quark-gluon plasma within the holographic correspondence for two arbitrary rotating parameters. For our analysis we consider dynamics of an open sting, with one endpoint attached at the spacetime boundary. The endpoint is fixed, so in the Boyer-Lindquist coordinates, this corresponds to a frame rotating-at-infinity, the endpoint rotates together with the quark-gluon plasma, while in the AdS coordinates, which are related to a static-at-infinity frame, the endpoint does not rotate, however the ansatz for the dynamical variables can be interpreted as a slow

motion of the string. According to the holographic dictionary, the conjugate momenta is related to a drag force and the total energy of the string corresponds to a thermal mass of a quark at rest.

We also consider an open string with both endpoints located on the boundary of the Kerr-AdS$_5$ spacetime and hanging down to the black hole horizon in the static-at-infinity frame. This string configuration can be associated with a light-like Wilson loop, which allows to estimate the jet-quenching parameter for a heavy quark in the rotating QGP.

The paper is organized as follows. First, in Section 2 we give a brief overview of an open string dynamics in the generic 5-dimensional gravitational background, we write down relations for the conjugate momenta and the total energy carried by the string. We also present the 5d Kerr-AdS black holes with two non-zero rotational parameters in Boyer-Lindquist coordinates (rotating-at-infinity), as well as in the so-called AdS coordinates (static at the boundary). In Section 3 we consider a straight static string, which holographically describes a static quark, and find the string energy, which is interpreted as a thermal static mass of the quark. We calculate the string energy for both frames of and analyze its high temperature limit. Section 4 is devoted to computing the holographic drag force using the string conjugate momenta in the Kerr-AdS black hole for both coordinate systems. In Section 5 the jet-quenching parameter is computed by means of a holographic light-like Wilson loop in Kerr-AdS$_5$ with one rotation parameter in the static-at-infinity frame. Finally Section 6 presents the conclusions. Some useful intermediate calculations are given in the Appendix. As indicated in some footnotes, this article is accompanied with SageMath notebooks.

2. The setup

2.1. Open string dynamics

Within the holographic approach a heavy quark is described by an open string moving in the black hole, which one endpoint is fixed to the spacetimes boundary. Thus, the momentum flowing from the boundary to the horizon is related to drag force applied on a heavy quark moving in the medium. The equations of motion for the string can be obtained from the Nambu-Goto action that reads as

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\sigma^0 d\sigma^1 \sqrt{-g}, \tag{2.1}$$

where σ^0, σ^1 are coordinates on the string worldsheet which parametrize the induced metric, $g = \det g_{\alpha\beta}$ is the determinant of the induced metric, given by

$$g_{\alpha\beta} = G_{MN}\partial_\alpha X^M \partial_\beta X^N, \tag{2.2}$$

with the 5d background metric G_{MN} and the embedding functions of the worldsheet with $X^\mu = X^\mu(\sigma)$, $\sqrt{\alpha'}$ is a fundamental string scale.

Varying the action (2.1) with respect to X^μ we get the equations of motion

$$\frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\, G_{\mu\nu}\, \partial^\alpha X^\nu\right) - \frac{1}{2}\partial_\mu G_{\rho\nu}\partial_\alpha X^\rho\, \partial^\alpha X^\nu = 0. \tag{2.3}$$

The conserved currents are given through the variational derivatives on $\partial_\alpha X^\mu$

$$\pi_\mu^\alpha \equiv -2\pi\alpha' \frac{\delta S}{\delta\partial_\alpha X^\mu} = \sqrt{-g}\, g^{\alpha\beta} G_{\mu\nu}(X)\partial_\beta X^\nu. \tag{2.4}$$

The energy density of the string is

$$\pi_t^{\sigma^0} = \sqrt{-g}\, g^{0\beta} G_{t\nu}(X)\, \partial_\beta X^\nu, \tag{2.5}$$

in what follows, we keep for the conjugate momenta $\pi^{\sigma^0} = \pi^0$, $\pi^{\sigma^1} = \pi^r$. Therefore the total energy of the string is given by

$$E = -\frac{1}{2\pi\alpha'} \int dr\, \pi_t^0. \tag{2.6}$$

Knowing the energy of the string one can calculate the mass of a single heavy quark and corrections to it. The energy (2.6) should be interpreted as the free energy of a static quark sitting in the $\mathcal{N} = 4$ SYM quark-gluon-plasma. Another quantity of our interest is the drag force acting on a heavy quark. The time-independent force on the string is

$$\frac{\partial p_\mu}{\partial \sigma^0} = -\frac{1}{2\pi\alpha'} \pi_\mu^r. \tag{2.7}$$

The force that acts on the string is calculated as follows

$$\frac{\partial p_\mu}{\partial \sigma^0} = -\frac{1}{2\pi\alpha'} \pi_\mu^r. \tag{2.8}$$

2.2. The 5d Kerr-AdS black hole in different coordinates

Here we briefly describe the 5d Kerr-AdS black hole with two non-zero rotational parameters. As it is known a rotating black hole in five dimensions can be characterized by the mass and two angular momentum invariants related to the numbers of Casimir invariants of $SO(4)$, therefore the solution has two rotational parameters.

The metric of the 5d Kerr-AdS solution [23] in terms of the Boyer-Lindquist coordinates is given by

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}d\psi\right)^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a\,dt - \frac{(r^2+a^2)}{\Xi_a}d\phi\right)^2$$

$$+ \frac{\Delta_\theta \cos^2\theta}{\rho^2}\left(b\,dt - \frac{(r^2+b^2)}{\Xi_b}d\psi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \tag{2.9}$$

$$+ \frac{(1+r^2\ell^2)}{r^2\rho^2}\left(ab\,dt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi\right)^2,$$

where M is the mass of the black hole, a, b are rotating parameters, constrained such that a^2, $b^2 \leq \ell^{-2}$ and we also denote

$$\Delta_r = \frac{1}{r^2}(r^2+a^2)(r^2+b^2)(1+r^2\ell^2) - 2M,$$

$$\Delta_\theta = (1 - a^2\ell^2\cos^2\theta - b^2\ell^2\sin^2\theta), \tag{2.10}$$

$$\rho^2 = (r^2 + a^2\cos^2\theta + b^2\sin^2\theta),$$

$$\Xi_a = (1 - a^2\ell^2), \quad \Xi_b = (1 - b^2\ell^2).$$

We note that for the spherical part of the metric we use Hopf coordinates, thus the angular coordinates are defined as $0 \leq \phi, \psi \leq 2\pi$, $0 \leq \theta \leq \pi/2$. The horizon position is defined as a

largest root r_+ to the equation $\Delta_r = 0$. So the solution (2.9) is defined on the region of the radial coordinate $r \in (r_+, +\infty)$, with the boundary of Kerr-AdS_5 at $r = +\infty$. Note, that the Boyer-Lindquist coordinates correspond to the rotating-at-infinity frame.

The conformal metric on the boundary with $r \to +\infty$ is represented by

$$
\begin{aligned}
ds^2_{BL} = &-dt^2 + \frac{2a\sin^2\theta}{\Xi_a}dt d\phi + \frac{2b\cos^2\theta}{\Xi_b}dt d\psi + \frac{\ell^2}{\Delta_\theta}d\theta^2 + \frac{\ell^2\sin^2\theta}{\Xi_a}d\phi^2 \\
&+ \frac{\ell^2\cos^2\theta}{\Xi_b}d\psi^2.
\end{aligned}
\tag{2.11}
$$

The Hawking temperature reads

$$
T_H = \frac{1}{2\pi}\left(r_+(1 + r_+^2\ell^2)\left(\frac{1}{r_+^2 + a^2} + \frac{1}{r_+^2 + b^2}\right) - \frac{1}{r_+}\right).
\tag{2.12}
$$

It is worth to be noted that it corresponds to the temperature of the dual field theory. We see that for a theory that lives on the boundary of the Kerr-AdS spacetime with the metric (2.11) and a temperature (2.12) the Ehrenfest–Tolman relation holds[1]

$$
\sqrt{|G_{00}|}T_H = \frac{1}{\beta} = \text{const.}
\tag{2.13}
$$

Plots with the dependence of the Hawking temperature on the horizon r_+ were presented in [20], where it was also shown that a Hawking-Page phase transition in the 5d Kerr-AdS black hole appears for a and b in the range of values from 0 to 0.25 with $\ell = 1$.

We note that in the Boyer-Lindquist coordinates, the metric is asymptotic to AdS_5 in a rotating frame, with the angular velocities

$$
\Omega_a^\infty = a\ell^2, \quad \Omega_b^\infty = b\ell^2.
\tag{2.14}
$$

Surprisingly, if one takes into account that the rotational parameters have the dimension of length, ℓ has the dimension of inverse length, we come that $\Omega_{a,b}^\infty$ are measured in fm^{-1} and coming to MeV, we get the following range for the angular velocities

$$
\Omega_{a,b}^\infty \approx 19.7 - 45.3 \quad \text{MeV},
\tag{2.15}
$$

that is in agreement with values of the angular velocity that is used in the hydrodynamical simulations [9].

The 5d Kerr-AdS black hole can be also represented in asymptotically AdS coordinates. Here it is convenient to take $\ell = 1$, then the coordinate transformations are

$$
\begin{aligned}
(1 - a^2)y^2\sin^2\Theta &= (r^2 + a^2)\sin^2\theta, \\
(1 - b^2)y^2\cos^2\Theta &= (r^2 + b^2)\cos^2\theta, \\
\Phi &= \phi + at, \\
\Psi &= \psi + bt, \\
T &= t.
\end{aligned}
\tag{2.16}
$$

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[1] We thank V. Braguta for pointing this.

For generic values of the rotational parameters a and b, we are not able to write down the metric in AdS coordinates. So, in what follows, we will use the approximate form from [26]. The line-element of the 5d Kerr-AdS black hole in the coordinates (2.16)-(2.16) near the spacetime boundary $y \to +\infty$ reads

$$
\begin{aligned}
ds^2 \simeq &-(1+y^2)dT^2 + \frac{dy^2}{1+y^2 - \frac{2M}{\Delta^2 y^2}} + y^2(d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2) \\
&+ \frac{2M}{\Delta^3 y^2}dT^2 + \frac{2Ma^2\sin^4\Theta}{\Delta^3 y^2}d\Phi^2 + \frac{2Mb^2\cos^4\Theta}{\Delta^3 y^2}d\Psi^2 - \\
&- \frac{4Ma\sin^2\Theta}{\Delta^3 y^2}dTd\Phi - \frac{4Mb\cos^2\Theta}{\Delta^3 y^2}dTd\Psi + \frac{4Mab\sin^2\Theta\cos^2\Theta}{\Delta^3 y^2}d\Phi d\Psi,
\end{aligned}
\tag{2.17}
$$

where

$$
\Delta = 1 - a^2\sin^2\Theta - b^2\cos^2\Theta.
\tag{2.18}
$$

Note, that the metric (2.17) with $M = 0$ describes the 5d AdS spacetime in the global coordinates.

The boundary of (2.17) doesn't rotate at infinity (static-at-infinity frame) and has the following form

$$
ds^2 = -dT^2 + d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2,
\tag{2.19}
$$

which describes a compactified Minkowski space and manifests $\mathbb{R} \times \mathbb{S}^3$ symmetry. It is worth to be noted that the boundary of the Kerr-AdS_5 in Boyer-Lindquist coordinates for small rotational parameters a and b is approximately (2.19). In [26] it was shown that the metrics of the boundary in BL coordinates and in static-at-the-boundary coordinates are conformally related as

$$
ds^2_{BL,bnd} = \frac{y^2}{r^2}ds^2_{AdS,bnd}.
\tag{2.20}
$$

So we can assume the Kerr-AdS black hole in BL coordinates is holographically dual to CFT at finite temperature in a rotating frame.

3. Energy of a static heavy quark

In this section we compute the thermal mass of a heavy quark and find how the rotation affects on it.

A static quark at rest in $\mathcal{N} = 4$ SYM quark-gluon plasma can be described by a straight string stretched from the spacetime boundary to the black hole horizon. The solutions to the string equations of motion that represent this configuration are just constants. For the string in the Kerr-AdS_5 background in rotating-at-infinity frame (2.9)-(2.10) and with the worldsheet parametrization $(\sigma^0, \sigma^1) = (t, r)$, it is given by

$$
\theta(r, t) = \theta_0, \quad \phi(t, r) = \phi_0, \quad \psi(t, r) = \psi_0.
\tag{3.1}
$$

To calculate the energy of the static quark one needs to find the energy of the string (3.1) defined by (2.6). To find this we need to calculate the conjugate momentum representing the energy density of a string. In the Boyer-Lindquist coordinates it has the following form

$$
\pi^0_t = \sqrt{-g}\left(g^{00}(G_{tt} + G_{t\phi}\dot\phi + G_{t\psi}\dot\psi) + g^{01}\left(G_{t\phi}\phi' + G_{t\psi}\psi'\right)\right).
\tag{3.2}
$$

Plugging (3.1) into (3.2) we get

$$\pi^0_t = \sqrt{-g}\, g^{00} G_{tt} = -\left(1 - \frac{\Delta_\theta}{\Delta_r}\left(a^2 \sin^2\theta + b^2 \cos^2\theta\right) - \frac{1}{\Delta_r}\frac{a^2 b^2(1+r^2\ell^2)}{r^2}\right). \quad (3.3)$$

Expanding (3.3) first in small a, then in small b and neglecting cross-terms of higher order we get:

$$\pi^0_t = -1 + \frac{a^2 \sin^2\theta + b^2 \cos^2\theta}{\ell^2 r^4 + r^2 - 2M}. \quad (3.4)$$

It's interesting to note that in (3.4) there is a contribution related to the rotational parameters. Another interesting thing, observed in (3.4), is that the expression has a dependence on the blackening factor $\ell^2 r^4 - 2M + r^2$ of the Schwarzchild-AdS black hole. This happens since the Kerr-AdS solution with small rotating parameters tends to the Schwarzchild-AdS one. Substituting (3.4) into (2.6) and integrating, we have

$$E = \frac{1}{2\pi\alpha'}\Bigg(r + \frac{(a^2 \sin^2\theta + b^2 \cos^2\theta)}{2\sqrt{M}\sqrt{8M\ell+1}}\Bigg(\sqrt{1+\sqrt{8M\ell+1}}\,\tanh^{-1}\Bigg(\frac{\sqrt{2}\ell r}{\sqrt{\sqrt{8M\ell+1}-1}}\Bigg)$$
$$- \sqrt{\sqrt{8M\ell+1}-1}\,\tan^{-1}\Bigg(\frac{\sqrt{2}\ell r}{\sqrt{1+\sqrt{8M\ell+1}}}\Bigg)\Bigg)\Bigg)\Bigg|^{r_m}_{r_H+\epsilon},$$

where we define the quark location by r_m.

At zero temperature limit, that corresponds to $M=0$, we get

$$E\Big|_{T=0} = \frac{1}{2\pi\alpha'}\Bigg(r + \sqrt{2}\ell\left(a^2 \sin^2\theta + b^2 \cos^2\theta\right)$$
$$\times \Bigg(\frac{1}{\sqrt{2}\ell r} - \frac{\tan^{-1}(\ell r)}{\sqrt{2}} - \frac{i\pi}{4\ell\sqrt{M}}\Bigg)\Bigg)\Bigg|^{r_m}_{r_H+\epsilon} = \qquad (3.5)$$
$$= \frac{1}{2\pi\alpha'}\left(r_m + \ell(a^2 \sin^2\theta + b^2 \cos^2\theta)\left(\frac{1}{\ell r_m} - \frac{1}{\varepsilon} - \tan^{-1}(\ell r_m)\right)\right),$$

where we use $\varepsilon = \ell\epsilon$ and $r_H = 0$ with $M = 0$. Note, that at $T = 0$ the renormalized energy equals to the (Lagrangian) mass m of the quark at zero temperature limit. So we have

$$E_{\text{ren}}\Big|_{T=0} = \frac{1}{2\pi\alpha'}\left(r_m + \ell(a^2 \sin^2\theta + b^2 \cos^2\theta)\left(\frac{1}{\ell r_m} - \tan^{-1}(\ell r_m)\right)\right) = m. \quad (3.6)$$

If one increases the temperature the energy takes the form

$$E = \frac{1}{2\pi\alpha'}\left(m - r_H - \left(\frac{\tanh^{-1}\left(\frac{r_H+\epsilon}{r_H}\right)}{r_H} - \ell\frac{\tan^{-1}\left(\frac{\ell(r_H+\epsilon)}{\sqrt{\ell^2 r_H^2+1}}\right)}{\sqrt{\ell^2 r_H^2+1}}\right)\frac{(a^2 \sin^2\theta + b^2 \cos^2\theta)}{(2\ell^2 r_H^2+1)}\right). \quad (3.7)$$

Therefore, the static thermal mass is given by

$$M_{\text{rest}} = m - \Delta m(T, a, b), \quad (3.8)$$

where

$$\Delta m(T, a, b) = \frac{\sqrt{\lambda}}{2\pi} \left(\frac{\pi T_H + \sqrt{\pi^2 T_H^2 - 2\ell^2}}{2\ell^2} + \frac{\ell^2 (a^2 \sin^2 \theta + b^2 \cos^2 \theta)}{\pi^2 T_H^2 + \pi T_H \sqrt{\pi^2 T_H^2 - 2\ell^2}} \right. \tag{3.9}$$

$$\times \left(\frac{\tanh^{-1}\left(1 + \frac{\epsilon 2\ell^2}{\pi T_H + \sqrt{\pi^2 T_H^2 - 2\ell^2}}\right)}{\pi T_H + \sqrt{\pi^2 T_H^2 - 2\ell^2}} - \frac{\tan^{-1}\left(\frac{(\pi T_H + \sqrt{\pi^2 T_H^2 - 2\ell^2} + 2\epsilon\ell^2)}{\sqrt{2}\sqrt{\pi^2 T_H^2 + \ell^2 + \pi T_H \sqrt{\pi^2 T_H^2 - 2\ell^2}}}\right)}{\sqrt{2}\sqrt{\pi^2 T_H^2 + \ell^2 + \pi T_H \sqrt{\pi^2 T_H^2 - 2\ell^2}}} \right) \Bigg) ,$$

with $\lambda = 1/\alpha'^2$.

Note, that in the limit of the large T_H the contribution from rotation is suppressed and comes to

$$\Delta m(T, a, b) \approx \frac{\sqrt{\lambda}}{2} \frac{T_H}{\ell^2}. \tag{3.10}$$

This is exactly that was found for the planar AdS black holes in [46].

Now we turn to the calculations in the AdS coordinates (non-rotating-at-infinity frame) (2.17). Using a similar worldsheet parametrization $(\sigma^0, \sigma^1) = (T, y)$ and considering a quark as an endpoint of a static string, we have for the conjugate momenta

$$\pi^0_t = G_{yy} G_{TT} = -1 + \frac{2M(a^2 \sin^2 \Theta + b^2 \cos^2 \Theta)}{\Delta(\Delta^2 y^2 + \Delta^2 y^4 - 2M)}, \tag{3.11}$$

where Δ is given by (2.18).

Taking into account that rotational parameters a and b are small, we get from (3.11)

$$\pi^0_t = -1 + \frac{2M(a^2 \sin^2 \Theta + b^2 \cos^2 \Theta)}{y^4 + y^2 - 2M}. \tag{3.12}$$

Correspondingly, for the energy (2.6) we have:

$$E = \frac{1}{2\pi\alpha'} \left(y + \frac{\sqrt{M}\left(a^2 \sin^2 \Theta + b^2 \cos^2 \Theta\right)}{\sqrt{8M + 1}} \left(\sqrt{1 + \sqrt{8M + 1}} \tanh^{-1}\left(\frac{\sqrt{2} y}{\sqrt{\sqrt{8M + 1} - 1}}\right) \right. \right.$$

$$\left. \left. - \sqrt{\sqrt{8M + 1} - 1} \tan^{-1}\left(\frac{\sqrt{2} y}{\sqrt{1 + \sqrt{8M + 1}}}\right) \right) \right) \Bigg|_{y_H + \epsilon}^{y_m}. \tag{3.13}$$

Note, that in (3.13) we again see terms related with rotation. However, in the limit of zero mass, that corresponds to the zero temperature, the contribution from rotation vanishes:

$$E|_{T=0} = \frac{1}{2\pi\alpha'} \left(y + 2 \left(\frac{M}{y} - M \tan^{-1} y - \frac{i\pi\sqrt{M}}{2\sqrt{2}} \right) \left(a^2 \sin^2 \Theta + b^2 \cos^2 \Theta\right) \right) \Bigg|_{y_H + \epsilon}^{y_m}$$

$$= y_m, \tag{3.14}$$

where we also take into account that y_H tends to zero as $M = 0$. This can be explained by that we consider the Kerr-AdS black hole in the non-rotating-at-infinity frame.

As in the case previous case this energy (3.14) corresponds to the mass of the quark at rest:

$$E|_{T=0} = \frac{1}{2\pi\alpha'} y_m = m. \tag{3.15}$$

Correspondingly, the total energy with a switched temperature taken into account is given by
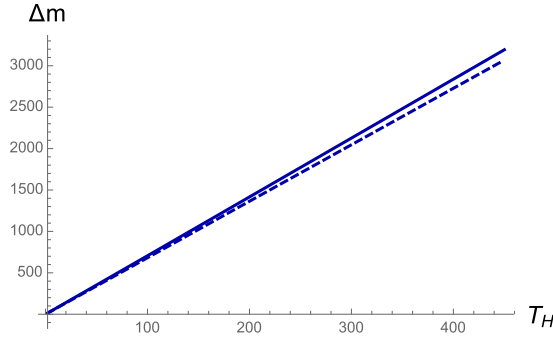
Fig. 1. The dependence of Δm (in the non-rotating-at-infinity-frame (3.9) and in the static-at-infinity-frame (3.18)) on the temperature T_H.

$$E = \frac{1}{2\pi\alpha'}\left(m - y_H - \left(\sqrt{(y_H^2+1)}\tanh^{-1}\left(1 + \frac{\epsilon}{y_H}\right) - y_H \tan^{-1}\left(\frac{y_H + \epsilon}{\sqrt{y_H^2+1}}\right)\right) \times \right.$$

$$\left. \times \frac{y_H(\sqrt{y_H^2+1})(a^2\sin^2\Theta + b^2\cos^2\Theta)}{2y_H^2+1}\right), \tag{3.16}$$

where we used the relation between y_H and M, namely, $M = \frac{y_H^2(y_H^2+1)}{2}$. Finally, we can represent the thermal mass of the quark at rest as

$$M_{\text{rest}} = m - \Delta m(T, a, b), \tag{3.17}$$

where m is given by (3.15) and

$$\Delta m(T, a, b) = \frac{\sqrt{\lambda}}{2\pi}\left(\frac{\pi T_H + \sqrt{\pi^2 T_H^2 - 2}}{2} + \right. \tag{3.18}$$

$$+ \left(\frac{2\tanh^{-1}\left(1 + \frac{2\epsilon}{\pi T_H + \sqrt{\pi^2 T_H^2 - 2}}\right)}{\pi T_H + \sqrt{\pi^2 T_H^2 - 2}} - \frac{\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\frac{\pi T_H + \sqrt{\pi^2 T_H^2 - 2 + 2\epsilon}}{\sqrt{\pi^2 T_H^2 + \pi T_H\sqrt{\pi^2 T_H^2 - 2} + 1}}\right)}{\sqrt{\pi^2 T_H^2 + \pi T_H\sqrt{\pi^2 T_H^2 - 2} + 1}}\right)$$

$$\left. \times \frac{\left((\pi^2 T_H^2 + \pi T_H\sqrt{\pi^2 T_H^2 - 2})^2 - 1\right)(a^2\sin^2\Theta + b^2\cos^2\Theta)}{4(\pi^2 T_H^2 + \pi T_H\sqrt{\pi^2 T_H^2 - 2})}\right).$$

From the relation (3.18) we see, that at high temperatures the rotation doesn't affect on the thermal mass of the static quark, moreover the approximate form of the correction looks like for the AdS black brane with a planar horizon [46] and represents a linear dependence on T_H

$$\Delta m(T, a, b) \approx \frac{\sqrt{\lambda}}{2} T_H. \tag{3.19}$$

The same dependence for $\Delta m(T, a, b)$ was observed for the Kerr-AdS_5 in the rotating-at-infinity frame (3.10). We compare the behaviors of $\Delta m(T, a, b)$ for different frames in Fig. 1.

4. Drag force acting on a heavy quark in 5d Kerr-AdS black hole

4.1. Calculations in Boyer-Lindquist coordinates

In this section we consider a trailing curved string in the 5d Kerr-AdS black hole in Boyer-Lindquist coordinates (2.9) with (2.10) (the rotating-at-infinity frame).[2] We parametrize the string worldsheet as $\sigma^0 = t$, $\sigma^1 = r$. For the embedding of the string worldsheet into the black hole background (2.9) we choose

$$\theta = \theta(t,r), \quad \phi = \phi(t,r), \quad \psi = \psi(t,r). \tag{4.1}$$

Owing to (4.1) the components of the induced metric (2.2) are

$$g_{00} = G_{tt} + G_{\theta\theta}\dot\theta^2 + G_{\phi\phi}\dot\phi^2 + G_{\psi\psi}\dot\psi^2 + 2\left(G_{t\phi}\dot\phi + G_{t\psi}\dot\psi + G_{\psi\phi}\dot\psi\dot\phi\right), \tag{4.2}$$

$$\begin{aligned} g_{01} &= g_{10} \\ &= G_{\theta\theta}\dot\theta\theta' + G_{\phi\phi}\dot\phi\phi' + G_{\psi\psi}\dot\psi\psi' + G_{t\phi}\phi' + G_{t\psi}\psi' + G_{\phi\psi}\left(\dot\phi\psi' + \phi'\dot\psi\right), \end{aligned} \tag{4.3}$$

$$g_{11} = G_{\theta\theta}\theta'^2 + G_{\phi\phi}\phi'^2 + G_{\psi\psi}\psi'^2 + G_{rr} + 2G_{\psi\phi}\psi'\phi', \tag{4.4}$$

where we denote $\dot{} = \frac{d}{dt}$ and $' = \frac{d}{dr}$.

The equations of motion (2.3) with (4.2)-(4.4) are difficult to work with, so we are going to consider the linearized equations, which we obtain using an expansion in small rotating parameters. We are looking for solutions to eqs. (2.3) in the form of the curved string solution given by the ansatz

$$\begin{aligned} \phi(t,r) &= \Phi_0 + a\ell^2 t + a\phi_1(r) + \mathcal{O}(a^2), \\ \theta(r) &= \Theta_0 + (a+b)^2\theta_1(r) + \mathcal{O}((a+b)^4), \\ \psi(t,r) &= \Psi_0 + b\ell^2 t + b\psi_1(r) + \mathcal{O}(b^2), \end{aligned} \tag{4.5}$$

where Θ_0, Φ_0, Ψ_0 are some constants, θ_1, ϕ_1, ψ_1 are some functions depending on the radial coordinate r. We recall that the string endpoint attached to the boundary is fixed and rotates with the thermal medium. So the terms $a\ell^2$, $b\ell^2$ can be associated to the components of the angular velocity of the boundary. To simplify calculations we also assume that the following relation between the angular parameters holds

$$b = na, \tag{4.6}$$

where n is some number thus the parameters a and b are of the same order.

The constraint (4.6) allows to reduce the calculations in terms of one parameter of expansion a. Plugging (4.5) with (4.6) into (4.2)-(4.4) the determinant of the induced metric takes the following form

$$\begin{aligned} -g &= \left(a^2\ell^2(G_{\phi\phi}\phi_1' + n^2 G_{\psi\psi}\psi_1') + a(G_{t\phi}\phi_1' + nG_{t\psi}\psi_1') + na^2\ell^2 G_{\phi\psi}\left(\psi_1' + \phi_1'\right)\right)^2 \\ &\quad - \left(G_{tt} + a^2\ell^4 G_{\phi\phi} + n^2 a^2\ell^4 G_{\psi\psi} + 2\ell^2\left(aG_{t\phi} + naG_{t\psi} + na^2\ell^2 G_{\psi\phi}\right)\right) \\ &\quad \times \left((1+n)^2 a^2 G_{\theta\theta}\theta_1'^2 + a^2 G_{\phi\phi}\phi_1'^2 + n^2 a^2 G_{\psi\psi}\psi_1'^2 + G_{rr} + 2na^2 G_{\psi\phi}\psi_1'\phi_1'\right), \end{aligned} \tag{4.7}$$

---

[2] See the SageMath notebook https://cocalc.com/share/6850015be0320058cadba1ce63e7a3eedf4eef89/Kerr-AdS-5D-string-b_na.ipynb for some details of the computations.

with the metric components (A.8)-(A.13).

Thanks to the ansatz (4.5) the equations of motion for θ, ϕ, ψ comes to the equations for θ_1, ϕ_1, ψ_1. One can see that ϕ_1 and ψ_1 are cyclic variables, the corresponding conservation laws for ϕ_1 and ψ_1 are given by

$$\phi_1 = \mathfrak{p} \int_{r_+}^{r} \frac{d\bar{r}}{\bar{r}^2 + \ell^2 \bar{r}^4 - 2M}, \qquad \psi_1 = \mathfrak{q} \int_{r_+}^{r} \frac{d\bar{r}}{\bar{r}^2 + \ell^2 \bar{r}^4 - 2M}, \tag{4.8}$$

where we used the conserved quantities $\mathfrak{p}, \mathfrak{q}$. Note, that we again observe the contribution of the Schwarzschild-AdS blackenning functions in (4.8). This appears because of small rotating parameters. Plugging (4.8) into (4.7) and doing expansion of the Lagrangian with (4.7) by a up to 4-th order we get the equation for θ_1, which can be represented as a first order differential equation, namely

$$
\begin{aligned}
\Upsilon' &+ \frac{2r + 4\ell^2 r^3}{r^2 + \ell^2 r^4 - 2M} \Upsilon \\
&+ \frac{1 + 4\ell^2 M(n^2 - 1) - \mathfrak{p}^2 + n^2(\mathfrak{q}^2 - 1) - 3(n^2 - 1)\ell^4 r^4}{2(1+n)^2 (r^2 + \ell^2 r^4 - 2M)^2} \sin(2\Theta_0) = 0,
\end{aligned}
\tag{4.9}
$$

where we define $\Upsilon = \theta_1'$.

The solution to (4.9) is represented in the following form

$$
\begin{aligned}
\theta_1' &= \frac{C_1}{r^4 f(r)} - \frac{3\ell^2(1 - n^2) r \sin(2\Theta_0)}{2(1+n)^2 r^4 f(r)} + \\
&+ \frac{(1 - n^2)\left(\ell^2 r_H^2 - 1\right)^2 - \mathfrak{p}^2 + n^2 \mathfrak{q}^2}{2(n+1)^2 f(r) r^4 (2\ell^2 r_H^2 + 1) r_H} \tanh^{-1}\left(\frac{r_H}{r}\right) \sin(2\Theta_0) \\
&+ \ell \frac{(1 - n^2)\left(\ell^2 r_H^2 + 2\right)^2 - \mathfrak{p}^2 + n^2 \mathfrak{q}^2}{2(n+1)^2 r^4 f(r)(2\ell^2 r_H^2 + 1)\sqrt{\ell^2 r_H^2 + 1}} \tan^{-1}\left(\frac{\ell r}{\sqrt{\ell^2 r_H^2 + 1}}\right) \sin(2\Theta_0),
\end{aligned}
\tag{4.10}
$$

where C_1 is a constant of integration, r_H is a horizon of the AdS-Schwarzchild black hole defined by (B.2) and the function $f(r)$ is the blackening function of the AdS-Schwarzchild black hole with a spherical horizon given by

$$f(r) = \ell^2 + \frac{1}{r^2} - \frac{2M}{r^4}. \tag{4.11}$$

We note that r_H is introduced in (4.10) recognizing some relations for the black hole mass (B.5)-(B.6). The appearance of $f(r)$ (4.11) and r_H in (4.9)-(4.10) can be explained the rotational parameters are small and the leading order is related to the non-rotating case. Taking into account $n = b/a$ we represent (4.10) as follows

$$
\begin{aligned}
\theta_1' &= \frac{C_1}{r^4 f(r)} - \frac{3\ell^2(a - b) r \sin(2\Theta_0)}{2(a+b) r^4 f(r)} + \\
&+ \frac{(a^2 - b^2)\left(\ell^2 r_H^2 - 1\right)^2 - a^2 \mathfrak{p}^2 + b^2 \mathfrak{q}^2}{2(a+b)^2 f(r) r^4 (2\ell^2 r_H^2 + 1) r_H} \tanh^{-1}\left(\frac{r_H}{r}\right) \sin(2\Theta_0)
\end{aligned}
\tag{4.12}
$$

$$
+ \ell \frac{\left(a^2 - b^2\right)\left(\ell^2 r_H^2 + 2\right)^2 - a^2 \mathfrak{p}^2 + b^2 \mathfrak{q}^2}{2(b+a)^2 r^4 f(r)(2\ell^2 r_H^2 + 1)\sqrt{\ell^2 r_H^2 + 1}} \tan^{-1}\left(\frac{\ell r}{\sqrt{\ell^2 r_H^2 + 1}}\right) \sin(2\Theta_0).
$$

The corresponding conjugate momenta (2.4) are calculated as

$$
\begin{aligned}
\pi_\theta^r &= f(r) r^4 \theta_1'(a+b)^2 + \mathcal{O}((a+b)^4), \\
\pi_\phi^r &= f(r) r^4 \phi_1' \sin^2(\Theta_0) a + \mathcal{O}(a^2), \\
\pi_\psi^r &= f(r) r^4 \psi_1' \cos^2(\Theta_0) b + \mathcal{O}(b^2).
\end{aligned}
\tag{4.13}
$$

Substituting (4.8) and (4.10) into (4.13) and expanding near the boundary $r \to +\infty$, we obtain for the conjugate momenta

$$
\pi_\theta^r = \left(\frac{2C_1}{\sin(2\Theta_0)} - 3r\ell^2(1-n^2) - \frac{3(1-n^2)}{r}\right.
\tag{4.14}
$$

$$
\left. + \frac{\ell\pi}{2(2\ell^2 r_H^2 + 1)} \frac{\left(1-n^2\right)\left(\ell^2 r_H^2 + 2\right)^2 - \mathfrak{p}^2 + n^2\mathfrak{q}^2}{\sqrt{\ell^2 r_H^2 + 1}}\right) \sin(2\Theta_0)\frac{a^2}{2} + \mathcal{O}((a+b)^4),
$$

$$
\pi_\phi^r = \mathfrak{p}\sin(\Theta_0)^2 a + \mathcal{O}(a^2), \quad \pi_\psi^r = \mathfrak{q}\cos(\Theta_0)^2 b + \mathcal{O}(b^2).
\tag{4.15}
$$

The components of the drag force for $a \neq b$ with $b = na$ are

$$
\frac{dp_\theta}{dt} = \left(-\frac{2C_1}{\sin(2\Theta_0)} + 3r\ell^2(1-n^2) + \frac{3(1-n^2)}{r}\right.
\tag{4.16}
$$

$$
\left. - \frac{\pi\ell}{2(2\ell^2 r_H^2 + 1)} \frac{\left(1-n^2\right)\left(\ell^2 r_H^2 + 2\right)^2 - \mathfrak{p}^2 + n^2\mathfrak{q}^2}{\sqrt{\ell^2 r_H^2 + 1}}\right) \frac{\sin(2\Theta_0)}{4\pi\alpha'} a^2 + \mathcal{O}((a+b)^4),
$$

$$
\frac{dp_\phi}{dt} = -\frac{1}{2\pi\alpha'} \mathfrak{p}\sin(\Theta_0)^2 a + \mathcal{O}(a^2), \quad \frac{dp_\psi}{dt} = -\frac{1}{2\pi\alpha'} \mathfrak{q}\cos(\Theta_0)^2 b + \mathcal{O}(b^2),
\tag{4.17}
$$

or by virtue of (4.12) we get for the θ-component

$$
\frac{dp_\theta}{dt} = \left(-\frac{2C_1}{\sin(2\Theta_0)} + 3r\ell^2\frac{a-b}{a+b} + \frac{3}{r}\frac{a-b}{a+b}\right.
$$

$$
\left. - \frac{\pi\ell}{2(2\ell^2 r_H^2 + 1)} \frac{\left(a^2 - b^2\right)\left(\ell^2 r_H^2 + 2\right)^2 - a^2\mathfrak{p}^2 + b^2\mathfrak{q}^2}{(a+b)^2\sqrt{\ell^2 r_H^2 + 1}}\right) \frac{\sin(2\Theta_0)}{4\pi\alpha'} (a+b)^2
\tag{4.18}
$$

$$
+ \mathcal{O}((a+b)^4).
$$

The linear divergent term (4.16) is associated with the mass of the heavy quark, which can be renormalized by a cut-off r_m. Taking $r_m = 2\pi\alpha' M_{\text{rest}}$ we get

$$
\frac{dp_\theta}{dt} = 3\frac{a-b}{a+b}\left(\ell^2 M_{\text{rest}} + \frac{1}{4\pi\alpha'}\left(\pi T_H + \sqrt{\pi^2 T_H^2 - 2\ell^2}\right)\right)\frac{(a+b)^2}{2}\sin(2\Theta_0) + \dots .
\tag{4.19}
$$

Taking into account (2.14) we can also represent the relation for the drag force (4.19) with the contributions from the rotational velocities

$$\frac{dp_\theta}{dt} = 3\frac{(\Omega_a^\infty)^2 - (\Omega_b^\infty)^2}{2\ell^2}\left(M_{\text{rest}} + \frac{\sqrt{\lambda}}{4\pi\ell^2}\left(\pi T_H + \sqrt{\pi^2 T_H^2 - 2\ell^2}\right)\right)\sin(2\Theta_0) + ... \quad .$$

$$(4.20)$$

Consider $b = 0$, this corresponds to the case with the one non-zero rotational parameter [20], we have for the components of the drag force:

$$\frac{dp_\theta}{dt} = 3\left(\ell^2 M_{\text{rest}} + \frac{1}{4\pi\alpha'}\left(\pi T_H + \sqrt{\pi^2 T_H^2 - 2\ell^2}\right)\right)\frac{a^2}{2}\sin(2\Theta_0) + ... \quad , \qquad (4.21)$$

$$\frac{dp_\phi}{dt} = -\frac{1}{2\pi\alpha'}\mathfrak{p}\sin(\Theta_0)^2 a + \mathcal{O}(a^2), \qquad \frac{dp_\psi}{dt} = 0. \qquad (4.22)$$

The relations (4.21)-(4.22) are also in agreement with the predictions from the 4d Kerr-AdS black hole [17]. However, it is worth to be noted that for two arbitrary non-zero rotational parameters the coefficient for the leading term is defined by the combination of a and b.

Another interesting particular case is the case of two equal rotational parameters, that corresponds to $n = 1$. From (4.19) we see that the component of the drag force $\frac{dp_\theta}{dt}$ just vanishes. This confirms the result obtained in [20] by using the metric with two equal parameters.

4.2. Calculations in asymptotically AdS coordinates

Now we turn to the calculations of the drag force in the 5d Kerr-AdS black hole in asymptotically AdS coordinates (2.17), i.e. non-rotating-at infinity frame.[3] For the coordinates on the string worldsheet we take the parametrization $\sigma^0 = T$, $\sigma^1 = y$, which is similar to the previous case.

The embedding is defined through

$$\Theta = \Theta(T, y), \quad \Phi = \Phi(T, y), \quad \Psi = \Psi(T, y). \qquad (4.23)$$

By virtue of (4.23), the induced metric (2.2) has the following components:

$$g_{00} = G_{TT} + G_{\Theta\Theta}\dot{\Theta}^2 + G_{\Phi\Phi}\dot{\Phi}^2 + G_{\Psi\Psi}\dot{\Psi}^2 + 2\left(G_{T\Phi}\dot{\Phi} + G_{T\Psi}\dot{\Psi} + G_{\Phi\Psi}\dot{\Phi}\dot{\Psi}\right), \quad (4.24)$$

$$g_{01} = G_{\Theta\Theta}\dot{\Theta}\Theta' + G_{\Phi\Phi}\dot{\Phi}\Phi' + G_{\Psi\Psi}\dot{\Psi}\Psi' + G_{T\Phi}\Phi' + G_{T\Psi}\Psi'$$
$$+ G_{\Phi\Psi}(\Phi'\dot{\Psi} + \dot{\Phi}\Psi'), \qquad (4.25)$$

$$g_{11} = G_{yy} + G_{\Theta\Theta}\Theta'^2 + G_{\Phi\Phi}\Phi'^2 + G_{\Psi\Psi}\Psi'^2 + 2G_{\Phi\Psi}\Phi'\Psi', \qquad (4.26)$$

where we define $\dot{} = \frac{d}{dt}$ and $' = \frac{d}{dr}$ and the components of the metric (2.17) are given by (A.14).

As in the Boyer-Lindquist coordinates the equations of motion (2.3) are complicated and we are going to look for the linearized solutions. Assuming that the rotational parameters are small and remembering the choice $\ell = 1$ in these coordinates, the curved string is given by the ansatz

$$\Phi(T, y) = \Phi_0 + aT + a\Phi_1(y) + \mathcal{O}(a^2),$$
$$\Theta(y) = \Theta_0 + \Theta_1(y)(a + b)^2 + \mathcal{O}((a + b)^4), \qquad (4.27)$$
$$\Psi(T, y) = \Psi_0 + bT + b\Psi_1(y) + \mathcal{O}(b^2),$$

---

[3] See the SageMath notebook https://cocalc.com/share/9e8e6273e8459a7ee654e124ec5ce50825096b2b/Kerr-AdS-5D-string-b_a-AdS.ipynb for some details of the computations.

where Φ_0, Θ_0, Ψ_0 are some constants. Comparing to the previous case, the boundary of the background doesn't rotate, however we suppose that the endpoint of the string slowly moves on the boundary.

With (4.27) the determinant of the induced metric comes to

$$
\begin{aligned}
-g = &\left((a^2 \sin^2 \Theta \Phi_1' + b^2 \cos^2 \Theta \Psi_1') \left(y^2 + \frac{2M}{\Delta^3 y^2}(a^2 \sin^2 \Theta + b^2 \cos^2 \Theta - 1) \right) \right)^2 \\
&- \left((a^2 \sin^2 \Theta + b^2 \cos^2 \Theta - 1) \left(y^2 + \frac{2M}{\Delta^3 y^2}(a^2 \sin^2 \Theta + b^2 \cos^2 \Theta - 1) \right) - 1 \right) \quad (4.28) \\
&\times \left(\frac{1}{1 + y^2 - \frac{2M}{\Delta^2 y^2}} + y^2((a+b)^4 \Theta_1'^2 + a^2 \sin^2 \Theta \Phi_1'^2 + b^2 \cos^2 \Theta \Psi_1'^2) + \right. \\
&\left. + \frac{2M}{\Delta^3 y^2}(a^2 \sin^2 \Theta \Phi_1' + b^2 \cos^2 \Theta \Psi_1')^2 \right),
\end{aligned}
$$

where Θ is given by (4.27). The form of the metric in the AdS coordinates (2.17) is simpler than in the Boyer-Lindquist coordinates (2.9), this leads to the relatively simple form of the induced metric (4.28). So, one doesn't need to put a constraint (4.6) we can do the expansion of the string Lagrangian first with respect to the parameter a and then with respect to b removing the cross-terms.

From (4.28) we see that the equations of motion are reduced to EOM for Θ_1, Φ_1 and Ψ_1. Moreover, for Φ_1 and Ψ_1 we have the conservation laws given by the following expressions:

$$
\Phi_1 = \mathfrak{p} \int_{y_+}^{y} \frac{d\bar{y}}{(\bar{y}^2 + \bar{y}^4 - 2M)}, \qquad \Psi_1 = \mathfrak{q} \int_{y_+}^{y} \frac{d\bar{y}}{(\bar{y}^2 + \bar{y}^4 - 2M)}, \quad (4.29)
$$

where $\mathfrak{p}$, $\mathfrak{q}$ are constants.

Plugging (4.29) into (4.28) and expanding the expression up to the 4th-order by a and the by b, removing the cross-terms of higher orders, we get the equation of motion for Θ_1'

$$
\Upsilon' + \Upsilon \frac{2y(1 + 2y^2)}{-2M + y^2 + y^4} + \frac{(y^4 - 4M)(a^2 - b^2) - a^2 \mathfrak{p}^2 + b^2 \mathfrak{q}^2}{2(a+b)^2 \left(-2M + y^2 + y^4 \right)^2} \sin 2\Theta_0 = 0, \quad (4.30)
$$

where $\Upsilon = \Theta_1'$. Eq. (4.30) has the following solution:

$$
\begin{aligned}
\Theta_1' = &\frac{C_1}{y^4 + y^2 - 2M} - \frac{y(a^2 - b^2)}{2(a+b)^2(y^2 + y^4 - 2M)} \sin(2\Theta_0) \quad (4.31) \\
&+ \frac{-(a^2 - b^2)y_H^2(y_H^2 + 2) - a^2 \mathfrak{p}^2 + b^2 \mathfrak{q}^2}{2y_H \left(2y_H^2 + 1 \right)(a+b)^2(y^2 + y^4 - 2M)} \tanh^{-1}\left(\frac{y_H}{y} \right) \sin(2\Theta_0) + \\
&+ \frac{(a^2 - b^2)(1 - y_H^4) - a^2 \mathfrak{p}^2 + b^2 \mathfrak{q}^2}{2\sqrt{y_H^2 + 1} \left(2y_H^2 + 1 \right)(a+b)^2(y^2 + y^4 - 2M)} \tan^{-1}\left(\frac{y}{\sqrt{y_H^2 + 1}} \right) \sin(2\Theta_0),
\end{aligned}
$$

where C_1 is a constant of integration.

The conjugate momenta (2.4) can be found as follows

$$
\begin{aligned}
\pi_\Theta^y &= (y^2 + y^4 - 2M)\Theta_1'(a+b)^2 + \mathcal{O}((a+b)^4), \\
\pi_\Phi^y &= (y^2 + y^4 - 2M)\sin^2(\Theta_0)\Phi_1' a + \mathcal{O}(a^2), \quad (4.32) \\
\pi_\Psi^y &= (y^2 + y^4 - 2M)\cos^2(\Theta_0)\Psi_1' b + \mathcal{O}(b^2).
\end{aligned}
$$

The drag force components (2.7) near the boundary of Kerr-AdS_5 $y \to +\infty$ are given by

$$\frac{dp_\Theta}{dt} = \Bigg(-\frac{2\tilde{C}_1}{\sin(2\Theta_0)} + \frac{y(a-b)}{(a+b)} + \frac{(a-b)}{(a+b)y} \tag{4.33}$$

$$-\frac{\pi}{2} \frac{(a^2-b^2)(1-y_H^4) - a^2\mathfrak{p}^2 + b^2\mathfrak{q}^2}{\sqrt{y_H^2+1}(2y_H^2+1)(a+b)^2} \Bigg) \frac{(a+b)^2}{4\pi\alpha'} \sin(2\Theta_0) + \mathcal{O}((a+b)^4),$$

$$\frac{dp_\Phi}{dt} = -\frac{\mathfrak{p}}{2\pi\alpha'} \sin^2(\Theta_0)a + \mathcal{O}(a^2), \quad \frac{dp_\Psi}{dt} = -\frac{\mathfrak{q}}{2\pi\alpha'} \cos^2(\Theta_0)b + \mathcal{O}(b^2). \tag{4.34}$$

As in the rotating-at-infinity frame we observe the divergent term with $y \to +\infty$ in (4.33), which can be associated to an infinite mass of the quark. Introducing a cut-off as $y_m = 2\pi\alpha' M_{\text{rest}}$ we get

$$\frac{dp_\Theta}{dt} \approx \frac{a-b}{a+b} \left(M_{\text{rest}} + \frac{1}{4\pi\alpha'} \left(\pi T_H + \sqrt{\pi^2 T_H^2 - 2} \right) \right) \frac{(a+b)^2}{2} \sin(2\Theta_0) + ..., \tag{4.35}$$

$$\frac{dp_\Phi}{dt} = -\frac{1}{2\pi\alpha'} \mathfrak{p} \sin(\Theta_0)^2 a + \mathcal{O}(a^2), \quad \frac{dp_\Psi}{dt} = 0. \tag{4.36}$$

Comparing (4.35) to (4.19) we see that the form of the relations for the drag force in the rotating-at-infinity and static frames is similar up to a number coefficient that appears probably because of using an approximate form of the metric in static-at-infinity frame.

Consider interesting particular cases, particularly, when we have just one non-zero rotational parameter, we again obtain a close relation but with another number coefficient, but the dependence on the rotational parameters is exactly the same as in (4.21)-(4.22). As for two equal rotational parameters $a = b$ we obtain that the drag force (4.35) is just zero, that matches with the results of the previous section and [20].

5. Jet quenching parameter

Now we turn to computation of the jet-quenching parameter following up the holographic prescription from [44]. The jet-quenching parameter $\hat{q}$ is related with the average of the light-like Wilson loop in the following form

$$W[C] = e^{-\frac{1}{4\sqrt{2}}\hat{q}L^- L^2}, \tag{5.1}$$

where C is a contour, L^- is an extension in a light-like direction and L is an extension in a transversal one. In the holographic approach $W[C]$ is equal to the classical string action stretched on the contour C on the boundary of the holographic background

$$W[C] = e^{-2S}. \tag{5.2}$$

We consider the Kerr-AdS black hole in static-at-the-boundary frame (2.17) with one non-zero rotational parameter $a \neq 0$. The metric reads

$$ds^2 \simeq -(1+y^2)dT^2 + \frac{dy^2}{1+y^2 - \frac{2M}{\Delta^2 y^2}} + y^2(d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2) \tag{5.3}$$

$$+ \frac{2M}{\Delta^3 y^2}\Big(dT^2 + a^2\sin^4\Theta d\Phi^2 - 2a\sin^2\Theta dTd\Phi\Big),$$

where $\Delta = 1 - a^2 \sin^2 \Theta$.

The light-cone coordinates are defined by

$$dx^+ = dT - ad\Phi, \quad dx^- = dT + ad\Phi. \tag{5.4}$$

$$dTd\Phi = \frac{1}{4a}\left((dx^-)^2 - (dx^+)^2\right), \quad dT = \frac{dx^+ + dx^-}{2}, \quad d\Phi = \frac{dx^- - dx^+}{2a}. \tag{5.5}$$

By owing of the coordinate transformation (5.4) the metric (5.3) takes the form[4]:

$$
\begin{aligned}
ds^2 \simeq &-\frac{1}{4}(dx^+)^2\left(1 + y^2 - \frac{y^2 \sin^2 \Theta}{a^2} - \frac{2M}{\Delta^3 y^2}(1 + \sin^2 \Theta)^2\right) \\
&- \frac{1}{4}(dx^-)^2\left(1 + y^2 - \frac{y^2 \sin^2 \Theta}{a^2} - \frac{2M}{\Delta^3 y^2}\cos^4 \Theta\right) \\
&- \frac{1}{2}dx^+dx^-\left(1 + y^2 + \frac{y^2 \sin^2 \Theta}{a^2} - \frac{2M}{\Delta^3 y^2}(1 - \sin^4 \Theta)\right) \\
&+ \frac{dy^2}{1 + y^2 - \frac{2M}{\Delta^2 y^2}} + y^2(d\Theta^2 + \cos^2 \Theta d\Psi^2).
\end{aligned}
\tag{5.6}
$$

The Wilson loop can be considered as an open string which both endpoints are attached to the boundary of the Kerr-AdS_5 background. The string worldsheet coordinates are given by

$$\sigma^0 = x^-, \quad \sigma^1 = \Psi. \tag{5.7}$$

Along Ψ we have length L and the lightcone direction has a length L^-. We suppose that

$$x^\mu = x^\mu(\sigma^1). \tag{5.8}$$

The Wilson loop lies at constant x^+ and Θ, then we have

$$\Theta(\sigma^1) = \text{const}, \quad x^+(\sigma^1) = \text{const}. \tag{5.9}$$

For the holographic coordinate y we impose the following constraint

$$y\left(\pm L/2\right) = \infty, \tag{5.10}$$

thus we have

$$y(\sigma^1) = y(-\sigma^1). \tag{5.11}$$

The Nambu-Goto action is given by (2.1) with the induced metric (2.2). By owing the parametrization (5.7)-(5.8) the induced metric components are

$$g_{00} = G_{x^-x^-}, \quad g_{01} = 0, \quad g_{11} = G_{yy}y'^2 + G_{\Psi\Psi}, \tag{5.12}$$

where we define by $A' = \frac{\partial A}{\partial \Psi}$.

---

[4] See the SageMath notebook https://cocalc.com/share/f8db4d9b694aa3a8b5ac7b72f83aa370a257c9f1/Kerr-AdS-5D-asymptotic_b_0.ipynb.

According to these calculations the Nambu-Goto action has the following view:

$$S = \frac{L^-}{2\pi\alpha'} \int\limits_{-L/2}^{L/2} d\Psi$$

$$\times \sqrt{\frac{1}{4}\left(y^2\cos^2\Theta + \frac{y'^2}{1+y^2-\frac{2M}{y^2\Delta^2}}\right)\left(1+y^2-\frac{y^2\sin^2\Theta}{a^2}-\frac{2M\cos^4\Theta}{\Delta^3 y^2}\right)}. \tag{5.13}$$

The first integral corresponding to (5.13) is

$$\mathcal{H} = \frac{2a^2 M\cos^4\Theta - \Delta^3 y^2(a^2 + (a^2-\sin^2\Theta)y^2)}{4a^2\Delta^3\mathcal{L}}\cos^2\Theta = -C, \tag{5.14}$$

where $\mathcal{L}$ is the Lagrangian corresponding to the action (5.13). Then the equation of motion is given by

$$y'^2 =$$

$$\frac{(2M-\Delta^2 y^2(1+y^2))(2a^2(2C^2\Delta^3 + 2M\cos^6\Theta) - \Delta^3\cos^2\Theta y^2(a^2+y^2(a^2-\sin^2\Theta)))}{4a^2 C^2\Delta^5}$$

$$\times\cos^2\Theta, \tag{5.15}$$

where $y' = \frac{\partial y}{\partial\sigma^1}$ and C is the constant energy of motion.

Plugging (5.15) back into (5.13) yields to

$$S = \frac{L^-}{2\pi\alpha'}\int\limits_{-L/2}^{L/2} d\Psi\frac{\cos^2\Theta}{4C}\left(y^2(1+y^2) - \frac{y^4\sin^2\Theta}{a^2} - \frac{2M\cos^4\Theta}{\Delta^3}\right). \tag{5.16}$$

Remembering that $d\Psi = \frac{dy}{y'}$ we come to the integration in terms of y, so we get for (5.16)

$$S = \frac{L^-}{\pi\alpha'}\int\limits_{y_H}^{\infty} dy \times \tag{5.17}$$

$$\frac{a\Delta^{5/2}\cos\Theta y^2\left(1+y^2-\frac{y^2\sin^2\Theta}{a^2}-\frac{2M\cos^4\Theta}{\Delta^3 y^2}\right)}{2\sqrt{(2M-\Delta^2 y^2(1+y^2))}\sqrt{2a^2(2C^2\Delta^3 + M\cos^6\Theta) - \Delta^3\cos^2\Theta y^2(a^2(1+y^2)-y^2\sin^2\Theta)}}.$$

The action (5.17) contains divergences on the spacetime boundary $y\to+\infty$. To renormalize this we have to subtract from (5.17) the mass of two single quarks, given by the term

$$S_0 = \frac{L^-}{\pi\alpha'}\int\limits_{y_H}^{\infty} dy\sqrt{|G_{x^-x^-}G_{yy}|} = \frac{L^-}{2\pi\alpha'}\int\limits_{y_H}^{\infty} dy\sqrt{\frac{1+y^2-\frac{y^2\sin^2\Theta}{a^2}-\frac{2M}{\Delta^3 y^2}\cos^4\Theta}{1+y^2-\frac{2M}{\Delta^2 y^2}}}. \tag{5.18}$$

Subtracting (5.18) from (5.17) we get the regularized Nambu-Goto action

$$S - S_0 = \frac{L^-}{\pi \alpha'} \int\limits_{y_H}^{\infty} dy \sqrt{\frac{1 + y^2 - \frac{y^2 \sin^2 \Theta}{a^2} - \frac{2M \cos^4 \Theta}{\Delta^3 y^2}}{1 + y^2 - \frac{2M}{\Delta^2 y^2}}} \times \tag{5.19}$$

$$\left(\frac{a \Delta^{3/2} \cos \Theta\, y \sqrt{1 + y^2 - \frac{y^2 \sin^2 \Theta}{a^2} - \frac{2M \cos^4 \Theta}{\Delta^3 y^2}}}{2 \sqrt{\Delta^3 \cos^2 \Theta\, y^2 (a^2(1 + y^2) - y^2 \sin^2 \Theta)) - 2a^2(2C^2\Delta^3 + M \cos^6 \Theta)}} - 1 \right).$$

Expanding in series assuming that M is large we get at first order

$$S^{reg} = S - S_0 \approx -\frac{L^-}{\pi \alpha'} \int\limits_{y_H}^{\infty} dy \frac{C^2 \Delta^{5/2} \sec^4 \Theta}{2M} + \mathcal{O}(M^2). \tag{5.20}$$

Integrating (5.20) in large M we get

$$S^{reg} \approx \sqrt{\lambda} \frac{L^-}{\pi} \frac{C^2 \Delta^{5/2} \sec^4 \Theta}{2M} y_H + \mathcal{O}(M^2), \tag{5.21}$$

where we define $\lambda = \frac{1}{\alpha'^2}$. For small a it has a simple form

$$S^{reg} \approx \sqrt{\lambda} \frac{L^-}{\pi} \frac{C^2 \sec^4 \Theta}{2M} y_H + \mathcal{O}(a^2). \tag{5.22}$$

To derive C we find the following relation for the length L from (5.15)

$$\frac{L}{2} = \int\limits_{y_H}^{\infty} dy \times \tag{5.23}$$

$$\frac{2aC\Delta^{5/2}}{\sqrt{(\Delta^2 y^2(1 + y^2) - 2M)(-2a^2(2C^2\Delta^3 + 2M \cos^6 \Theta) + \Delta^3 \cos^2 \Theta\, y^2(a^2 + y^2(a^2 - \sin^2 \Theta)))} \cos \Theta}.$$

At hight temperature limit ($M \to \infty$) eq. (5.23) takes the form

$$\frac{L}{2} \approx -\int\limits_{y_H}^{\infty} dy \frac{C \Delta^{5/2} \sec^4 \Theta}{M} + \mathcal{O}(M^2) \approx \frac{C \Delta^{5/2} \sec^4 \Theta}{M} y_H + \mathcal{O}(M^2), \tag{5.24}$$

or for small a

$$\frac{L}{2C} = -\frac{\sec^4 \Theta}{M} y_H + \mathcal{O}(a^2). \tag{5.25}$$

Then from (5.22) and (5.25) we find that the regularized action is

$$2S_I^{reg} = \frac{\sqrt{\lambda}}{\pi} \frac{L^- L^2 M \cos^4 \Theta}{4 y_H}. \tag{5.26}$$

Taking into account that $M = \frac{y_H^2 (y_H^2 + 1)}{2}$ in terms of T_H, related to y_H as $y_H = (\pi T_H + \sqrt{\pi^2 T_H^2 - 2})/2$ we have

$$2S_I^{reg} = \frac{\sqrt{\lambda}}{32\pi} L^- L^2 (\sqrt{\pi^2 T_H^2 - 2} + \pi T_H)(\pi T_H(\sqrt{\pi^2 T_H^2 - 2} + \pi T_H) + 1) \cos^4 \Theta.$$

(5.27)

By virtue of (5.1), (5.2) and (5.27) we find that the jet-quenching parameter is given

$$\hat{q} = \sqrt{\lambda} \frac{\left(\sqrt{\pi^2 T_H^2 - 2} + \pi T_H\right) \left(\pi T_H \left(\sqrt{\pi^2 T_H^2 - 2} + \pi T_H\right) + 1\right) \cos^4 \Theta}{4\sqrt{2}\pi},$$

(5.28)

which at high temperature takes the form

$$\hat{q} = \frac{\sqrt{\lambda}}{\sqrt{2}} \pi^2 T_H^3 \cos^4 \Theta.$$

(5.29)

In (5.29) we observe that the jet-quenching parameter depends on the temperature as T_H^3 and linearly on $\sqrt{\lambda}$ as in the case of the planar black brane [44].

6. Summary and discussion

In the present paper, we have investigated the energy loss of a heavy quark in a rotating quark-gluon plasma within the holographic framework. As a holographic dual we have considered the 5d Kerr-AdS black hole with two arbitrary rotational parameters. The Kerr-AdS black hole is justified for the description of the QGP at high temperature, where it is believed that the conformal symmetry is restored. Using the holographic prescription we have calculated the energy of a static quark in both the rotating- and static-at-infinity frames under the assumption that the rotational parameters are small. These relations generalize our result [20] obtained for a single rotational parameter (in the BL coordinates). We show that for small values of the rotational parameters the mass of a static quark in the rotating- and static-at-infinity frames, are similar. We have shown that at high temperature the contribution from the rotation in both coordinate systems is suppressed, thus the dependence of the energy on the temperature turns to be the linear relation $\frac{\sqrt{\lambda}}{2} T_H$ like for the case of the planar black brane [46]. It is interesting to note that in BL coordinates (the rotating-at-infinity frame), for zero temperature the Lagrangian mass of the static quark has a contribution from the rotational parameters, while in non-rotating-at-infinity frame (AdS coordinates) it does not.

We have also presented the general formulae for the holographic drag forces acting on a heavy quark for the case of two arbitrary rotational parameters. In the holographic description the heavy quark can be considered through a string hanging towards the black hole horizon with a fixed endpoint attached to the boundary of the Kerr-AdS$_5$ background. We have considered the rotating-at-infinity frame of the Kerr-AdS$_5$ with the string endpoint fixed on the boundary, so the string rotates together with the QGP and the static-at-infinity frame with a slowly moving endpoint of the string. We have shown that the results for the drag force are equivalent in both frames and have a linear dependence on the temperature T_H. The only difference is a numerical factor, which comes from the fact that we have used the approximated form of the metric for the static-at-infinity frame. We have seen that the drag force has a contribution from the difference of squares of the angular velocities as in the hydrodynamical calculations for the pressure gradient [20]. The drag force relations are also in agreement with the particular cases, i.e. a single non-zero rotational parameter and two equal rotational parameters, obtained in the previous paper [20]. For the case of two equal rotational parameters, which includes the extremal case, we have

found that the drag force vanishes. This was also observed in [20] using the hydrodynamical calculations. Note that for the case of a single rotational parameter in the rotating-at-infinity frame, the relation for the drag force matches with that one from [17].

In the case of the static-at-infinity frame and a single rotational parameter a, we have estimated the jet-quenching parameter and found that the leading term (for a small a) has the same dependence on $\sqrt{\lambda}$ and T_H as in the case of a planar AdS black brane [44].

As an extension for future work, it will be interesting to study the jet-quenching parameter more carefully in both the rotating- and static-at-infinity frames, to compute and analyze the contributions from the rotation of a higher order.

Another natural direction is to generalize the results for the energy losses on the case of the rotating charged AdS black holes [59], which correspond to a conformal plasma with non-zero chemical potential. It would be important to trace and analyze the influence of the chemical potential and the second non-zero rotational parameters on the transport properties of the QGP. It will be also of interest to compare results with those for the rotating quark in $\mathcal{N} = 4$ SYM (static) quark-gluon plasma at finite chemical potential from [60].

CRediT authorship contribution statement

Anastasia A. Golubtsova: Conceptualization, Investigation, Validation, Writing – review & editing. **Eric Gourgoulhon:** Investigation, Methodology, Validation, Writing – review & editing. **Marina K. Usova:** Investigation, Validation, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

Appendix A. Metric components

A.1. Boyer-Lindquist coordinates

The components of the metric (2.9) in general are

$$G_{tt} = \frac{1}{\rho^2}\left(-\Delta_r + \Delta_\theta\left(a^2\sin^2\theta + b^2\cos^2\theta\right) + \frac{a^2b^2(1+r^2\ell^2)}{r^2}\right), \tag{A.1}$$

$$G_{rr} = \frac{\rho^2}{\Delta_r}, \quad G_{\theta\theta} = \frac{\rho^2}{\Delta_\theta}, \tag{A.2}$$

$$G_{\phi\phi} = \frac{\sin^2\theta}{\rho^2\,\Xi_a^2}\left(-a^2\sin^2\theta\,\Delta_r + (r^2+a^2)^2(\Delta_\theta + \frac{b^2\sin^2\theta}{r^2}(1+r^2\ell^2))\right), \tag{A.3}$$

$$G_{\psi\psi} = \frac{\cos^2\theta}{\rho^2 \Xi_b^2} \left(-b^2 \cos^2\theta \Delta_r + (r^2 + b^2)^2 (\Delta_\theta + \frac{a^2 \cos^2\theta}{r^2}(1 + r^2\ell^2)) \right), \tag{A.4}$$

$$G_{t\phi} = G_{\phi t} = \frac{a\sin^2\theta}{\rho^2 \Xi_a} \left(\Delta_r - (r^2 + a^2)(\Delta_\theta + \frac{b^2}{r^2}(1 + r^2\ell^2)) \right), \tag{A.5}$$

$$G_{t\psi} = G_{\psi t} = \frac{b\cos^2\theta}{\rho^2 \Xi_b} \left(\Delta_r - (r^2 + b^2)(\Delta_\theta + \frac{a^2}{r^2}(1 + r^2\ell^2)) \right), \tag{A.6}$$

$$G_{\phi\psi} = G_{\psi\phi} = \frac{abM\sin^2(2\theta)}{2\rho^2 \Xi_a \Xi_b}. \tag{A.7}$$

If we suppose that $b = an$ then the metric components (A.1)-(A.7) are transformed to

$$G_{\phi\phi} = \frac{\sin^2\theta}{\rho^2 \Xi_a^2} \left(-a^2\sin^2\theta\Delta_r + (r^2 + a^2)^2(1 - a^2\ell^2\cos^2\theta + \frac{1}{r^2}n^2a^2\sin^2\theta) \right), \tag{A.8}$$

$$G_{\psi\psi} = \frac{\cos^2\theta}{\rho^2 \Xi_b^2} \left(-n^2a^2\cos^2\theta\Delta_r + (r^2 + n^2a^2)^2(1 - a^2n^2\ell^2\sin^2\theta + \frac{1}{r^2}a^2\cos^2\theta) \right), \tag{A.9}$$

$$G_{t\phi} = G_{\phi t} = \frac{a\sin^2\theta}{\rho^2 \Xi_a} \left(\Delta_r - (r^2 + a^2)(1 + \frac{n^2a^2}{r^2} - a^2\ell^2\cos^2\theta(1 - n^2)) \right), \tag{A.10}$$

$$G_{t\psi} = G_{\psi t} = \frac{na\cos^2\theta}{\rho^2 \Xi_b} \left(\Delta_r - (r^2 + n^2a^2)(1 + \frac{a^2}{r^2} + a^2\ell^2\sin^2\theta(1 - n^2)) \right), \tag{A.11}$$

$$G_{\phi\psi} = G_{\psi\phi} = \frac{na^2M\sin^2(2\theta)}{2\rho^2 \Xi_a \Xi_b}, \quad G_{rr} = \frac{\rho^2}{\Delta_r}, \quad G_{\theta\theta} = \frac{\rho^2}{\Delta_\theta}, \tag{A.12}$$

$$G_{tt} = \frac{1}{\rho^2} \left(-\Delta_r + a^2\Delta_\theta \left(\sin^2\theta + n^2\cos^2\theta \right) + \frac{n^2a^4(1 + r^2l^2)}{r^2} \right). \tag{A.13}$$

A.2. Asymptotically AdS coordinates

The components of the metric (2.17):

$$G_{TT} = \frac{2M}{\Delta^3 y^2} - (1 + y^2), \quad G_{yy} = \frac{1}{1 + y^2 - \frac{2M}{\Delta^2 y^2}}, \quad G_{\Theta\Theta} = y^2, \tag{A.14}$$

$$G_{\Phi\Phi} = y^2\sin^2\Theta + \frac{2Ma^2\sin^4\Theta}{\Delta^3 y^2}, \quad G_{\Psi\Psi} = y^2\cos^2\Theta + \frac{2Mb^2\cos^4\Theta}{\Delta^3 y^2}, \tag{A.15}$$

$$G_{T\Phi} = -\frac{2Ma\sin^2\Theta}{\Delta^3 y^2}, \quad G_{T\Psi} = -\frac{2Mb\cos^2\Theta}{\Delta^3 y^2}, \quad G_{\Phi\Psi} = \frac{2Mab\sin^2\Theta\cos^2\Theta}{\Delta^3 y^2}. \tag{A.16}$$

A.3. Light-cone coordinates

The metric components in light-cone coordinates are given by

$$G_{x^-x^-} = -\frac{1}{4}\left(1 + y^2 - \frac{y^2\sin^2\Theta}{a^2} - \frac{2M}{\Delta^3 y^2}\cos^4\Theta\right), \tag{A.17}$$

$$G_{x^+x^+} = -\frac{1}{4}\left(1 + y^2 - \frac{y^2 \sin^2\Theta}{a^2} - \frac{2M}{\Delta^3 y^2}(1 + \sin^2\Theta)^2\right), \tag{A.18}$$

$$G_{x^+x^-} = -\frac{1}{4}\left(1 + y^2 + \frac{y^2 \sin^2\Theta}{a^2} - \frac{2M}{\Delta^3 y^2}(1 - \sin^4\Theta)\right), \tag{A.19}$$

$$G_{yy} = \frac{1}{1 + y^2 - \frac{2M}{\Delta^2 y^2}}, \quad G_{\Theta\Theta} = y^2, \quad G_{\Psi\Psi} = y^2 \cos^2\Theta. \tag{A.20}$$

Appendix B. The limits of 5d Kerr-AdS metric. Special cases

For zero parameters a and b the metric (2.9) is simplified:

$$ds^2 = -r^2 f(r)dt^2 + r^2\left(d\theta^2 + \frac{dr^2}{r^4 f(r)} + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\right), \tag{B.1}$$

where $f(r) = l^2 + \frac{1}{r^2} - \frac{2M}{r^4}$ is the blackening function. This is nothing but the 5d Schwarzchild-AdS metric with the horizon located at

$$r_H = \frac{\sqrt{\sqrt{1 + 8\ell^2 M} - 1}}{\sqrt{2}\ell}, \tag{B.2}$$

which is calculated as the largest root of $f(r) = 0$. So for the solution (B.1) the radial coordinate runs as $r \in (r_H, +\infty)$. The Hawking temperature is given by

$$T_H = \frac{4r_H^2 \ell^2 + 2}{4\pi r_H}, \tag{B.3}$$

the location of the horizon is defined through the temperature as

$$r_H = \frac{1}{2\ell^2}\left(\pi T_H + \sqrt{\pi^2 T_H^2 - 2\ell^2}\right). \tag{B.4}$$

From (B.2) one can write down some useful relations

$$2\ell^2 r_H^2 = \sqrt{1 + 8\ell^2 M} - 1, \quad 2\ell^2 r_H^2 + 1 = \sqrt{1 + 8\ell^2 M}, \tag{B.5}$$

$$1 + 4\ell^2 M - \sqrt{1 + 8\ell^2 M} = 2\ell^4 r_H^4, \quad 4(\ell^2 r_H^2 + 1)^2 = (\sqrt{1 + 8\ell^2 M} + 1)^2. \tag{B.6}$$

We note that $r_H = 0$ if

$$M = 0, \tag{B.7}$$

in this case the metric (2.9) is just the 5d AdS spacetime in the global coordinates.

It is also instructive to show that the metric (2.17) with $a = b = 0$ has $\Delta = 1$ comes to the following form

$$ds^2 \simeq -(1 + y^2 - \frac{2M}{y^2})dT^2 + \frac{dy^2}{1 + y^2 - \frac{2M}{y^2}} + y^2(d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2). \tag{B.8}$$

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