Trapping Horizons as Inner Boundary Conditions for Black Hole Spacetimes

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1. New horizons

2. Application to 3+1 numerical relativity
Outline

1. New horizons

2. Application to 3+1 numerical relativity
Classical definition of a black hole

black hole:

\[ B := M - J^- (I^+) \]

i.e. the region of spacetime where light rays cannot escape to infinity

- \( M \) = asymptotically flat manifold
- \( I^+ \) = future null infinity
- \( J^- (I^+) \) = causal past of \( I^+ \)

event horizon: \( H := \partial J^- (I^+) \)

(boundary of \( J^- (I^+) \))

\( H \) smooth \( \implies \) \( H \) null hypersurface
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event horizon: \( \mathcal{H} := \dot{J}^{-}(I^+) \)
(boundary of \( J^{-}(I^+) \))

\( \mathcal{H} \) smooth \( \Rightarrow \) \( \mathcal{H} \) null hypersurface
This is a highly non-local definition!

The determination of the boundary of $J^-(\mathcal{I}^+)$ requires the knowledge of the entire future null infinity. Moreover this is not locally linked with the notion of strong gravitational field:

Example of event horizon in a flat region of spacetime:
Vaidya metric, describing incoming radiation from infinity:

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2dv dr + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$

with

- $m(v) = 0$ for $v < 0$
- $dm/dv > 0$ for $0 \leq v \leq v_0$
- $m(v) = M_0$ for $v > v_0$

[Ashtekar & Krishnan, LRR 7, 10 (2004)]
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$\Rightarrow$ no local physical experiment whatsoever can locate the event horizon

[Ashtekar & Krishnan, LRR 7, 10 (2004)]
Another non-local feature: teleological nature of event horizons

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To deal with black holes as physical objects, a local definition would be desirable.
Recently a **new paradigm** appeared in the theoretical approach of black holes: instead of *event horizons*, black holes are described by

- **trapping horizons** (Hayward 1994)
- **isolated horizons** (Ashtekar et al. 1999)
- **dynamical horizons** (Ashtekar and Krishnan 2002)

All these concepts are **local** and are based on the notion of **trapped surfaces**

**Motivations:** quantum gravity, numerical relativity
Consider a spacelike 2-surface $S$ (induced metric: $q$)
What is a trapped surface?

1/ Expansion of a surface along a normal vector field

1. Consider a spacelike 2-surface $S$ (induced metric: $q$)
2. Take a vector field $v$ defined on $S$ and normal to $S$ at each point

\[ \theta(v) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\delta A)' - \delta A = L_v \ln \sqrt{q} = q_{\mu\nu} \nabla^\mu v^\nu \]
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5. This defines a new surface $S'$ (Lie dragging)
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Do the same for each point in $S$, keeping the value of $\varepsilon$ fixed

This defines a new surface $S'$ (Lie dragging)

At each point, the expansion of $S$ along $v$ is defined from the relative change in the area element $\delta A$:

$$\theta(v) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{\delta A' - \delta A}{\delta A} = \mathcal{L}_v \ln \sqrt{q} = q^{\mu\nu} \nabla_\mu v_\nu$$
What is a trapped surface?

2/ The definition

\( S \): closed (i.e. compact without boundary) spacelike 2-dimensional surface embedded in spacetime \((\mathcal{M}, g)\)
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Being spacelike, \( S \) lies outside the light cone.
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2/ The definition

$S$: closed (i.e. compact without boundary) spacelike 2-dimensional surface embedded in spacetime $(\mathcal{M}, g)$

Being spacelike, $S$ lies outside the light cone

$\exists$ two future-directed null directions orthogonal to $S$:

- $\ell$ = outgoing, expansion $\theta^{(\ell)}$
- $k$ = ingoing, expansion $\theta^{(k)}$

In flat space, $\theta^{(k)} < 0$ and $\theta^{(\ell)} > 0$

$S$ is trapped $\iff \theta^{(k)} < 0$ and $\theta^{(\ell)} < 0$

$S$ is marginally trapped $\iff \theta^{(k)} < 0$ and $\theta^{(\ell)} = 0$
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$\mathcal{S}$ : closed (i.e. compact without boundary) spacelike 2-dimensional surface embedded in spacetime $(\mathcal{M}, g)$

Being spacelike, $\mathcal{S}$ lies outside the light cone

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$\mathcal{S}$ is trapped $\iff$ $\theta^{(k)} < 0$ and $\theta^{(\ell)} < 0$ [Penrose 1965]

$\mathcal{S}$ is marginally trapped $\iff$ $\theta^{(k)} < 0$ and $\theta^{(\ell)} = 0$
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**trapped surface** = **local** concept characterizing very strong gravitational fields
A closed spacelike 2-surface $S$ is said to be **outer trapped** (resp. **marginally outer trapped (MOTS)**) iff [Hawking & Ellis 1973]

- the notions of *interior* and *exterior* of $S$ can be defined (for instance spacetime asymptotically flat) $\Rightarrow \ell$ is chosen to be the *outgoing* null normal and $k$ to be the *ingoing* one
- $\theta(\ell) < 0$ (resp. $\theta(\ell) = 0$)
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**Proposition** [Hawking & Ellis 1973]: $\mathcal{A}$ smooth $\Rightarrow \mathcal{A}$ is a MOTS
Proposition [Penrose (1965)]: 
provided that the weak energy condition holds, 
\(\exists \) a trapped surface \( S \implies \exists \) a singularity in \((\mathcal{M}, g)\) (in the form of a future inextendible null geodesic)

Proposition [Hawking & Ellis (1973)]: 
provided that the cosmic censorship conjecture holds, 
\(\exists \) a trapped surface \( S \implies \exists \) a black hole \( B \) and \( S \subset B \)
Local definitions of “black holes”

A hypersurface $\mathcal{H}$ of $(\mathcal{M}, g)$ is said to be

- a **future outer trapping horizon (FOTH)** iff
  
  (i) $\mathcal{H}$ foliated by marginally trapped 2-surfaces
      ($\theta^{(k)} < 0$ and $\theta^{(\ell)} = 0$)
  
  (ii) $\mathcal{L}_k \theta^{(\ell)} < 0$ (locally outermost trapped surf.)

  [Hayward, PRD 49, 6467 (1994)]
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- a **dynamical horizon (DH)** iff
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- a **non-expanding horizon (NEH)** iff
  (i) $\mathcal{H}$ is null (null normal $\ell$)
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- an **isolated horizon (IH)** iff
  (i) $\mathcal{H}$ is a non-expanding horizon
  (ii) $\mathcal{H}$’s full geometry is not evolving along the
  null generators: $[\mathcal{L}_\ell, \hat{\nabla}] = 0$
  [Ashtekar, Beetle & Fairhurst, CQG 16, L1 (1999)]
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- **a dynamical horizon (DH)** iff
  (i) $\mathcal{H}$ foliated by marginally trapped 2-surfaces
  (ii) $\mathcal{H}$ spacelike
  [Ashtekar & Krishnan, PRL 89 261101 (2002)]

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Dynamics of these new horizons

The trapping horizons and dynamical horizons have their own dynamics, ruled by Einstein equations. In particular, one can establish for them

- existence and (partial) uniqueness theorems
  
  [Andersson, Mars & Simon, PRL 95, 111102 (2005)],

- first and second laws of black hole mechanics
  
  [Ashtekar & Krishnan, PRD 68, 104030 (2003)], [Hayward, PRD 70, 104027 (2004)]

- a viscous fluid bubble analogy ("membrane paradigm" as for the event horizon), leading to a Navier-Stokes-like equation and a positive bulk viscosity (event horizon = negative bulk viscosity)
  
  [Gourgoulhon, PRD 72, 104007 (2005)], [Gourgoulhon & Jaramillo, gr-qc/0607050]

Outline

1 New horizons

2 Application to 3+1 numerical relativity
The basic idea

Use the concepts of trapping/dynamical horizons in the very construction of a 3+1 black hole spacetime

... and not as a *posteriori* analysis tools as in e.g. [Dreyer, Krishnan, Shoemaker & Schnetter, PRD 67, 024018 (2003)], [Schnetter, Krishnan & Beyer, gr-qc/0604015]

Related previous proposals (*prior to the introduction of trapping/dynamical horizons*): use of a MOTS (apparent horizon) as inner boundary conditions for excision [Thornburg, CQG 4, 1119 (1987)], [Eardley, PRD 57, 2299 (1998)]

Already used for initial data (IH) (*cf. M. Ansorg’s and H. Pfeiffer’s talks*)
Excision

**Framework:** 3+1 formalism: spacetime slicing by a family $\left(\Sigma_t\right)_{t \in \mathbb{R}}$ of spacelike hypersurfaces

**Excision method** to deal with black holes: excise from the numerical domain a region containing the singularity

![Diagram of spacetime slicing and excision](image)
Excision

**Framework:** 3+1 formalism: spacetime slicing by a family \((\Sigma_t)_{t \in \mathbb{R}}\) of spacelike hypersurfaces

**Excision method** to deal with black holes: excise from the numerical domain a region containing the singularity

Provided that the excised region is located within the even horizon, the choice of it does not affect the exterior spacetime
In the constrained scheme based on Dirac gauge + maximal slicing [Bonazzola, Gourgoulhon, Grandclément & Novak, PRD 70, 104007 (2004)] (cf. J. Novak’s talk), boundary conditions are required for the elliptic equations governing

- the conformal factor $\Psi$
- the lapse function $N$
- the shift vector $\beta$

NB: no need of boundary conditions for the metric potentials $h^{ij} := \tilde{\gamma}^{ij} - f^{ij}$

[I. Cordero Carrión (2006)]
Choose the excision boundary $S_t$ to be a **marginally trapped surface** for each time $t$

The tube $\mathcal{H} = \bigcup_{t \in \mathbb{R}} S_t$

is then generically a smooth **trapping horizon**

[Andersson, Mars & Simon, PRL 95, 111102 (2005)]

- geometrically well defined excision boundary
- ensures $S_t$ is located inside the event horizon
- easy to implement with spherical coordinates and spectral methods
Non-uniqueness of trapping horizons

Different 3+1 slicings may lead to different trapping horizons

$NB$: uniqueness in spherical symmetry
Geometrical setup

Hypersurface $\Sigma_t$:
- induced metric $\gamma$ (positive definite); associated connection $D$
- future directed timelike unit normal $n$
- extrinsic curvature $K : K_{\alpha\beta} = -\nabla_{\mu} n_{\alpha} \gamma^{\mu}_{\beta}$
- lapse function $N : n = -N dt$
Geometrical setup

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- lapse function $N : n = -N dt$

2-surface $S_t$ :
- induced metric $q$ (positive definite); associated connection $D$
- normal vector pairs (basis of $T_p(S_t)^\perp$):
  - orthonormal basis $(n, s)$, where $s$ is the outgoing spacelike unit normal to $S_t$ in $\Sigma_t$
  - null basis $(\ell, k)$ (not unique: $\ell \mapsto \ell' = \lambda \ell$, $k \mapsto k' = \mu k$)
- extrinsic curvature, as a hypersurface of $\Sigma_t$, $H : H_{\alpha\beta} = D_\mu s_\alpha q^\mu_{\beta}$
Vector field $h$ on $\mathcal{H}$ defined by

1. $h$ is tangent to $\mathcal{H}$
2. $h$ is orthogonal to $S_t$
3. $\mathcal{L}_h = h^\mu \partial_\mu t = \langle dt, h \rangle = 1$

NB: (iii) $\implies$ the 2-surfaces $S_t$ are Lie-dragged by $h$

$h \in T_p(S_t)^\perp = \text{Vect}(n, s)$ and can be decomposed as $h = Nn + bs$
Norm of $h$ and type of $\mathcal{H}$

Definition: $C := \frac{1}{2} h \cdot h = \frac{1}{2} (b^2 - N^2)$

$\mathcal{H}$ is spacelike (DH) $\iff$ $h$ is spacelike $\iff$ $C > 0$ $\iff$ $b > N$

$\mathcal{H}$ is null (IH) $\iff$ $h$ is null $\iff$ $C = 0$ $\iff$ $b = N$

$\mathcal{H}$ is timelike $\iff$ $h$ is timelike $\iff$ $C < 0$ $\iff$ $b < N$. 

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NFNR, Golm, 21 July 2006
Null basis associated with $h$

The vectors $\ell := \frac{1}{2}(b + N)(n + s)$ and $k := \frac{1}{b + N}(n - s)$ are the unique pair of null vectors normal to $S_t$ such that $\ell \cdot k = -1$ and $h = \ell - Ck$. 

![Diagram](image-url)
Spatial coordinates

Coordinates \((x^i)_{i \in \{1,2,3\}}\) on \(\Sigma_t\) \(\Rightarrow\) defines the shift vector \(\beta\):

\[
\partial_t = Nn + \beta
\]

2+1 orthogonal decomposition of the shift with respect to \(S_t\):

\[
\beta = \beta^\perp s - V \quad \text{with} \quad s \cdot V = 0.
\]

The coordinates \((t, x^i)\) are comoving w.r.t. \(\mathcal{H}\) iff there exists a function \(f\) not depending on \(t\) and such that

\[
\forall p = (t, x^1, x^2, x^3) \in \mathcal{M}, \; p \in \mathcal{H} \iff f(x^1, x^2, x^3) = 0
\]

Special case: adapted coordinates: \(f = f(x^1)\)

Coordinates \((t, x^i)\) comoving w.r.t. \(\mathcal{H}\) \(\iff\) \(\partial_t\) tangent to \(\mathcal{H}\)

\[
\iff \beta^\perp = b \iff h = \partial_t + V
\]
Condition $\theta(\ell) = 0$ on $S_t$

Preliminary: 2+1 orthogonal decomposition of the extrinsic curvature of $\Sigma_t$:

$$K = -\sigma^{(n)} - \frac{1}{2}\theta^{(n)}q + s \otimes L + L \otimes s + K(s, s)s \otimes s$$

with $\text{tr} \sigma^{(n)} = 0$ (shear of $S_t$ along $n$) and $L := K(s, q)$
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$$K = -\sigma^{(n)} - \frac{1}{2} \theta^{(n)} q + s \otimes L + L \otimes s + K(s, s) s \otimes s$$

- part tangent to $S_t$
- mixed part
- normal part

with $\text{tr} \sigma^{(n)} = 0$ (shear of $S_t$ along $n$) and $L := K(s, \bar{q})$

One has $\theta(\ell) = \frac{1}{2} (b + N) \left[ \theta^{(n)} + \theta^{(s)} \right]$.

Since $\theta^{(n)} = K(s, s) - K$ (see above) and $\theta^{(s)} = H = D \cdot s$,

we get $\theta(\ell) = \frac{1}{2} (b + N) [D \cdot s + K(s, s) - K]$.

Hence the well known marginally trapped surface condition:

$$\theta(\ell) = 0 \iff D \cdot s + K(s, s) - K = 0$$

which yields, in a conformal decomposition ($\gamma = \Psi^4 \tilde{\gamma}$),

$$4\tilde{s} \cdot \tilde{D} \Psi + K(\tilde{s}, \tilde{s}) \Psi^{-2} - K \Psi^2 + \tilde{D} \cdot \tilde{s} = 0 \quad (1)$$
Condition $\mathcal{L}_h \theta^{(\ell)} = 0$ on $S_t$

i.e. not only $S_t$ is a marginally trapped surface at time $t$, but remains marginally trapped at time $t + \delta t$:

Thanks to Einstein equation, the condition $\mathcal{L}_h \theta^{(\ell)} = 0$, along with $\theta^{(\ell)} = 0$, is equivalent to [Eardley, PRD 57, 2299 (1998)]

$$-D_a D^a (b - N) - 2L^a D_a (b - N) + A (b - N) = B (b + N) \quad (2)$$

with

$$L_a := K_{ij} s^i q^j_a$$

$$A := \frac{1}{2} \mathcal{R} - D_a L^a - L_a L^a - 4\pi T_{\mu\nu} (n^\mu + s^\mu)(n^\nu - s^\nu)$$

$\mathcal{R}$ : Ricci scalar of the metric $q$ on $S_t$

$$B := \frac{1}{2} \hat{\sigma}_{ab} \hat{\sigma}^{ab} + 4\pi T_{\mu\nu} (n^\mu + s^\mu)(n^\nu + s^\nu)$$

$$\hat{\sigma}_{ab} := H_{ab} - \frac{1}{2} H q_{ab} + \sigma^{(n)}_{ab}$$
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$$\hat{\sigma}_{ab} := H_{ab} - \frac{1}{2} H q_{ab} + \sigma^{(n)}_{ab}$$

**Remark:** for an isolated horizon, $B = 0$ and the solution to Eq. (2) is $b - N = 0$, which, in comoving coordinates w.r.t. $\mathcal{H}$, translates to $\beta^\perp = N$ (cf. H. Pfeiffer’s talk)
BC for the tangential part of the shift vector

Recall: \( \beta = \beta^\perp s - V \) and in comoving coord. w.r.t. \( \mathcal{H} \), \( \beta^\perp = b \) & \( h = \partial_t + V \)

shear tensor \( \sigma^{(h)} \) of the surface \( S_t \) along its evolution \( = \) traceless part of the deformation tensor of \( S_t \): \( \mathcal{L}_h q =: \theta^{(h)} q + 2\sigma^{(h)} \)

In comoving coord. \( 2\sigma_{ab}^{(h)} = \frac{\partial q_{ab}}{\partial t} - \frac{\partial}{\partial t} \ln \sqrt{q} \, q_{ab} + \mathcal{D}_a V_b + \mathcal{D}_b V_a - \mathcal{D}_c V^c q_{ab} \)
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shear tensor $\sigma^{(h)}$ of the surface $S_t$ along its evolution $= \text{traceless part of the deformation tensor of } S_t$: $\mathcal{L}_h q =: \theta^{(h)} q + 2\sigma^{(h)}$

In comoving coord. $2\sigma_{ab}^{(h)} = \frac{\partial q_{ab}}{\partial t} - \frac{\partial}{\partial t} \ln \sqrt{q} q_{ab} + D_a V_b + D_b V_a - D_c V^c q_{ab}$

Demand: the components of the metric on $S_t$ vary as less as possible, i.e. vary only to reflect the expansion of $S_t$:

$$\frac{\partial q_{ab}}{\partial t} - \frac{\partial}{\partial t} \ln \sqrt{q} q_{ab} = 0 \iff D_a V_b + D_b V_a - D_c V^c q_{ab} = 2\sigma_{ab}^{(h)}$$ (3)

$\sigma_{ab}^{(h)}$ being determined via the evolution equation

$$\mathcal{L}_h \sigma^{(h)} = -\bar{q}^* \text{Weyl}(\ell, \ldots, \ell, \ldots) - C^2 \bar{q}^* \text{Weyl}(k, \ldots, k, \ldots) - 8\pi C \left[ \bar{q}^* T - \frac{1}{2} (q : T) q \right] + \cdots$$

Remark: for an isolated horizon, $\sigma^{(h)} = 0$ and Eq. (3) says that $V$ must be a conformal Killing vector of $(S_t, q)$ (cf. H. Pfeiffer’s talk)
**Choice of the 3+1 slicing**

**NB1:** The trapping horizon condition by itself does specify the value of the lapse $N$, but only of the combination of $b - N$ and $b + N$ which appears in Eq. (2). Given an initial marginally trapped surface $S_0 \subset \Sigma_0$, the choice of $b$ and $N$ on $S_0$ determines a unique trapping horizon among all those which intersects $\Sigma_0$ in $S_0$.

**NB2:** The 3+1 slicing $(\Sigma_t)_{t \in \mathbb{R}}$ is determined by
(i) a condition “in the bulk” (e.g. maximal slicing)
(ii) the value of the lapse on $S_t$
In other words, (i) is not sufficient to specify uniquely the 3+1 slicing.
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Having chosen (i), one can use (ii) to select a trapping horizon $H$ with “good” properties.
For instance, we can demand that the area $A(t)$ of each section $S_t$ is maximal [Gourgoulhon & Jaramillo, gr-qc/0607050]. This translate into

$$b - N = \alpha D \cdot s, \quad \alpha = \text{const.}$$

Other choices, based on the convexity of $A(t)$, are possible [gr-qc/0607050]
The trapping horizon conditions + some coordinate choice lead to 5 equations to set the values of the 5 fields $\Psi, N, \beta^1, \beta^2, \beta^3$ at the excision surface $S_t$:

- **trapping horizon conditions:**
  - $\theta^{(\ell)} = 0 \implies \boxed{\Psi}$ [Eq. (1)]
  - $\mathcal{L}_h \theta^{(\ell)} = 0 \implies f_1(b - N, b + N)$ [Eq. (2)]

- **coordinate choice:**
  - comoving coordinates w.r.t. $\mathcal{H}$: $\boxed{\beta^\perp = b}$
  - traceless part of $\frac{\partial q_{ab}}{\partial t} = 0 \implies \boxed{V}$ [Eq. (3)]
  - choice of slicing/lapse $\implies f_2(b - N, b + N)$ [Eq. (4)]
Application to 3+1 numerical relativity

http://www.luth.obspm.fr/IHP06/

General Relativity Trimester

Gravitational Waves, Relativistic Astrophysics and Cosmology

Centre Emile Borel

INSTITUT HENRI POINCARÉ

Paris
18 September - 15 December 2006

Organizers:
Thibault Damour (IHES) and Nathalie Deruelle (IHES)

Co-organizers:
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This 3-month Programme will offer a series of courses designed for graduate students, post-docs, as well as scientists implied in experimental projects in General Relativity. It will also offer an array of shorter courses and seminars to bring students, post-docs and interested researchers to the level required to follow and contribute to some of the latest developments in General Relativity.

contact: grihp@ihp.jussieu.fr

Workshop

From Geometry to Numerics
IHP, Paris
20-24 November 2006

Ultra-preliminary list of speakers:
M. Ansorg
Y. Choquet-Bruhat
P. Chrusciel
J. Frauendiener
S. Hayward (tbc)
J. Isenberg
B. Krishnan
V. Moncrief
N. O’ Murchadha
D. Pollack
J. York (tbc)