Black hole thermodynamics

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Sadi Carnot's Legacy
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Outline

1. Black holes in relativistic gravity
2. The laws of classical black hole dynamics
   - The zeroth law
   - The first law
   - The second law
   - A third law?
3. Hawking radiation and black hole thermodynamics
4. Final remarks

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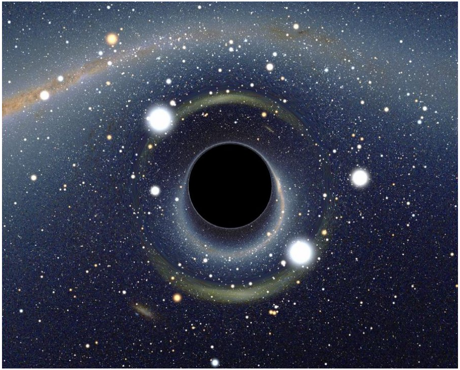
What is a black hole?

A layperson definition

A **black hole** is a localized region of spacetime from which no particle, be it massive or massless (photon), can escape to an infinitely remote region.



[A. Riazuelo, IJMPD **28**, 1950042 (2019)]

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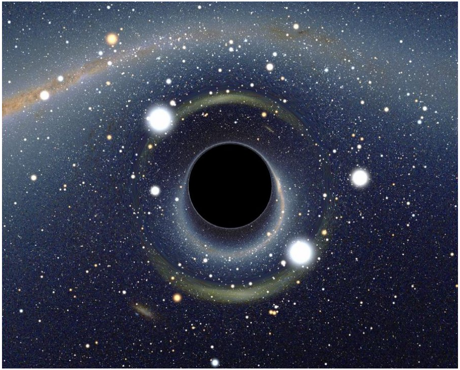

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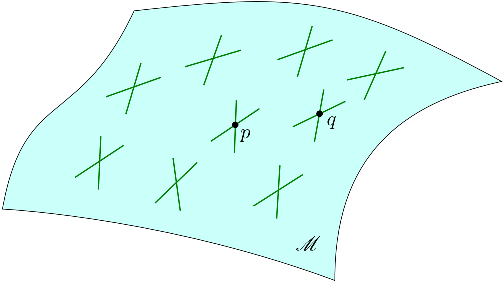
The (immaterial) boundary of the black hole region is called the **event horizon**

Relativistic spacetime



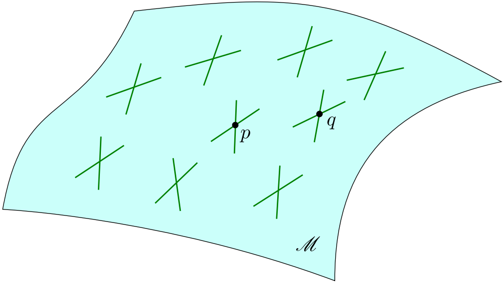
Spacetime $(\mathscr{M}, \boldsymbol{g})$

- $\mathscr{M}$ = 4-dimensional smooth manifold
- $\boldsymbol{g}$ = **metric tensor**: field of symmetric bilinear forms, of signature $(-,+,+,+)$

$\Longrightarrow$ pseudo-scalar product

$$\mathrm{d}s^2 := \boldsymbol{g}(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{x}) = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

Relativistic spacetime



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Example 1: Minkowski spacetime *(special relativity)*

$\mathscr{M} = \mathbb{R}^4$; $\quad \mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)$

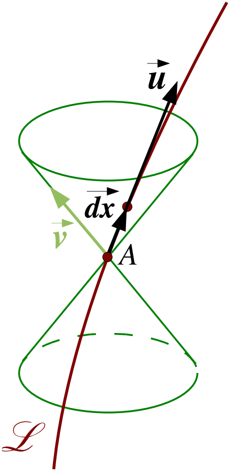
Example 2: Schwarzschild spacetime *(static black hole)*

$\mathscr{M} = \mathbb{R}^2 \times \mathbb{S}^2$;

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)$$

Physical meaning of the metric tensor

$\mathrm{sign}(\boldsymbol{g}) = (-,+,+,+) \Longrightarrow$ **null cones** $\Longrightarrow$ spacetime's **causal structure**



Proper time

Proper time τ of an observer = *length* measured via $\boldsymbol{g}$ along the observer's worldline $\mathscr{L}$

$$\mathrm{d}\tau = \frac{1}{c}\sqrt{-\boldsymbol{g}(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{x})}$$

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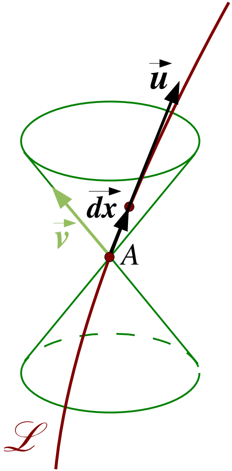

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Light rays

Wordlines of photons = *null geodesics* of $\boldsymbol{g}$.

General relativity

$(\mathscr{M}, \boldsymbol{g})$ is ruled by general relativity $\iff$ $\boldsymbol{g}$ obeys **Einstein's equation:**

$$\boldsymbol{R} - \frac{1}{2} R \, \boldsymbol{g} + \Lambda \, \boldsymbol{g} = \frac{8\pi G}{c^4} \boldsymbol{T}$$

where

- $\boldsymbol{R} := \mathrm{Ric}(\boldsymbol{g})$, Ricci tensor: $R_{\alpha\beta} = \mathrm{Riem}(\boldsymbol{g})^{\mu}{}_{\alpha\mu\beta}$
- $\mathrm{Riem}(\boldsymbol{g})$: Riemann curvature tensor
- $R := g^{\mu\nu} R_{\mu\nu}$, Ricci scalar
- Λ: cosmological constant
- $\boldsymbol{T}$: energy-momentum tensor of matter/fields

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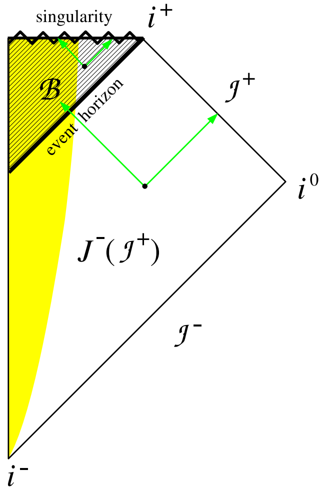
Many alternative theories of gravity are being considered, mainly in the framework of testing general relativity or explaining the dark energy mystery
A large class of them are the *scalar-tensor theories*: $\boldsymbol{g} \Longrightarrow (\boldsymbol{g}, \phi)$

Black hole definition in a metric theory of gravity



Carter-Penrose diagram

Spacetime $(\mathscr{M}, \boldsymbol{g})$ with **asymptotic infinity** $\mathscr{I}$: region "$r \to +\infty$" modeled as the boundary $\mathscr{I}$ of a larger spacetime $(\tilde{\mathscr{M}}, \tilde{\boldsymbol{g}})$ such that

$\tilde{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}, \ \ \tilde{\boldsymbol{g}} = \Omega^2 \boldsymbol{g}, \ \ \Omega|_{\mathscr{I}} = 0$

(*conformal completion* [Penrose (1963)])

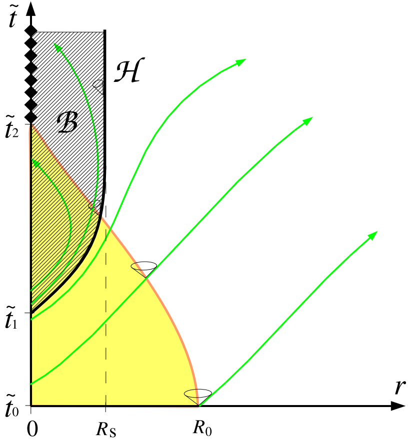
Future null infinity $\mathscr{I}^+$: part of $\mathscr{I}$ that can be reached by future-directed causal curves

The **black hole region** is the complement of the causal past of $\mathscr{I}^+$:

$$\mathscr{B} := \mathscr{M} \setminus J^-(\mathscr{I}^+)$$

The **event horizon** is the boundary of the black hole region: $\mathscr{H} := \partial\mathscr{B}$

Black hole definition in a metric theory of gravity



Spacetime diagram based on $(\tilde{t}, r)$ coordinates

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The event horizon is a null hypersurface

The black hole event horizon $\mathscr{H}$ is a null hypersurface of $(\mathscr{M}, \boldsymbol{g})$

A **hypersurface** of a 4-dimensional manifold $\mathscr{M}$ is a submanifold of $\mathscr{M}$ of dimension 3 (codimension 1).

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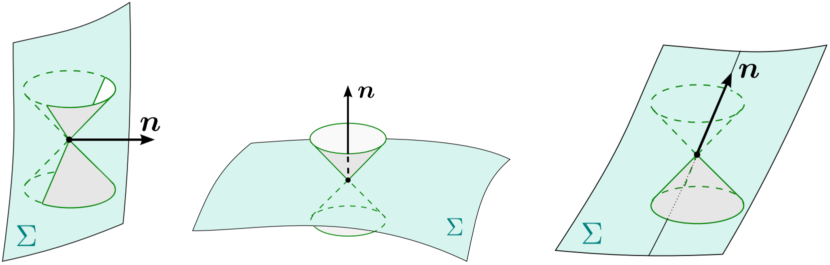
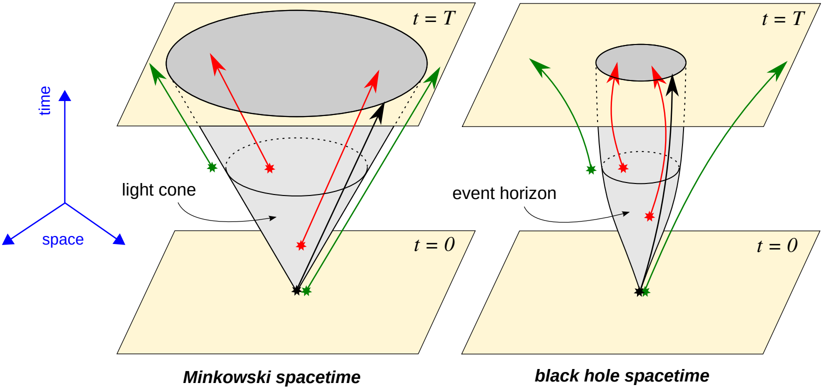
Locally, a hypersurface Σ can be of one of 3 types ($\boldsymbol{n}$ = normal to Σ):



Σ timelike	Σ spacelike	Σ null
$\boldsymbol{g}\|_\Sigma$ Lorentzian	$\boldsymbol{g}\|_\Sigma$ Riemannian	$\boldsymbol{g}\|_\Sigma$ degenerate
$\boldsymbol{n}$ spacelike	$\boldsymbol{n}$ timelike	$\boldsymbol{n}$ null (and tangent to Σ)

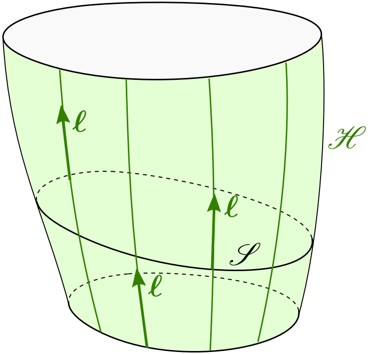
The event horizon is a null hypersurface

As a null hypersurface, the event horizon $\mathscr{H}$ is similar to a light cone in Minkowski spacetime:



Both hypersurfaces are **one-way membranes**; the event horizon is distinguished by its bounded spatial extension

Geodesic generators of a null hypersurface



cross-section $\mathscr{S}$: 2-surface of $\mathscr{H}$ intersected at most once by a given null generator

Let $\mathscr{H}$ be a null hypersurface:

- $\mathscr{H}$ is ruled by a family of null geodesics, called the **null generators of** $\mathscr{H}$;
- any vector $\boldsymbol{\ell}$ normal to $\mathscr{H}$ is tangent to a null generator
  $\Longrightarrow$ $\boldsymbol{\ell}$ obeys the pregeodesic equation:

$$\boldsymbol{\nabla}_{\boldsymbol{\ell}}\boldsymbol{\ell} = \kappa\boldsymbol{\ell} \iff \ell^\mu \nabla_\mu \ell^\alpha = \kappa\, \ell^\alpha$$

$\boldsymbol{\nabla}$: covariant derivative associated to $\boldsymbol{g}$
κ: **non-affinity coefficient** of $\boldsymbol{\ell}$
$\kappa = 0 \iff$ the generator parameter λ such that $\boldsymbol{\ell} = \dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\lambda}$ is an *affine parameter*

The no-hair theorem

Uniqueness theorem ("no-hair")
(Dorochkevitch, Novikov & Zeldovitch 1965, Israel 1967, Carter 1971, Hawking 1972, Robinson 1975)

Within 4-dimensional general relativity and modulo some "reasonable" hypotheses, any *stationary* black hole is a Kerr-Newman black hole, which is entirely described by only three numbers:

- its mass M
- its angular momentum J
- its electric charge Q

Special cases:

- $Q = 0$: Kerr BH (1963)
- $J = 0$: Reissner-Nordström BH (1916)
- $Q = 0,\ J = 0$: Schwarzschild BH (1915)

$\Longrightarrow$ *"A black hole has no hair"* (John A. Wheeler)

The Kerr black hole

Kerr solution to the vacuum Einstein equation (1963)

Expression in Boyer-Lindquist coordinates (t, r, θ, φ):

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\rho^2}\,\mathrm{d}t\,\mathrm{d}\varphi + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\varphi^2$$

where $a := J/M$, $\rho^2 := r^2 + a^2\cos^2\theta$, $\Delta := r^2 - 2Mr + a^2$

→ spacetime manifold: $\mathscr{M} = \mathbb{R}^2 \times \mathbb{S}^2 \setminus \{r = 0 \;\&\; \theta = \pi/2\}$;
NB: $r \in (-\infty, \infty)$

→ describes a rotating black hole with the event horizon $\mathscr{H}$ located at $r = r_{\mathscr{H}} := M + \sqrt{M^2 - a^2}$

Physical meaning of the parameters M and J

- Mass M: *not* a measure of the "amount of matter" inside the black hole, but rather a *characteristic of the external gravitational field* $\rightarrow$ measurable from the orbital period of a test particle in remote circular orbit around the black hole *(Kepler's third law)*

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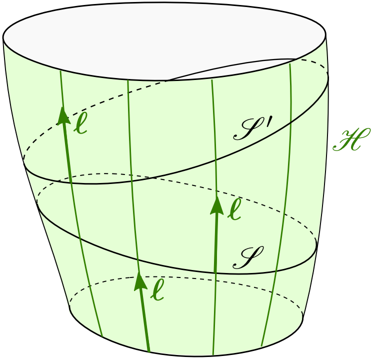
  $\rightarrow$ measurable from the orbital period of a test particle in remote circular orbit around the black hole *(Kepler's third law)*
- Angular momentum J: characterizes the *gravito-magnetic* part of the gravitational field
  $\rightarrow$ measurable from the precession of a gyroscope orbiting the black hole *(Lense-Thirring effect)*

Black hole area

No well-defined concept of *radius* to evaluate the "size" of a black hole $\Longrightarrow$ on the contrary, the *area* is well defined and locally measurable



Each cross-section $\mathscr{S}$ of the event horizon $\mathscr{H}$ is a **spacelike surface**: the metric $\boldsymbol{q}$ induced by $\boldsymbol{g}$ on $\mathscr{S}$ is Riemannian $(\mathrm{sign}\,\boldsymbol{q} = (+,+))$

The **area of** $\mathscr{S}$ is

$$A(\mathscr{S}) = \int_{\mathscr{S}} \sqrt{q}\, \mathrm{d}y^1 \mathrm{d}y^2,$$

where (y^1, y^2) are coordinates on $\mathscr{S}$ and $q := \det(q_{ab})$

For a *stationary* black hole, the area $A(\mathscr{S})$ is independent of the choice of the cross-section $\mathscr{S} \Longrightarrow$ **area** A **of the black hole**

Black hole area

Example: Kerr black hole

$$A = 8\pi(M^2 + \sqrt{M^4 - J^2})$$

Schwarzschild limit ($J = 0$): $A = 16\pi M^2$

Restoring the G and c's $\Longrightarrow A = 16\pi \left(\frac{GM}{c^2}\right)^2$

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Restoring the G and c's $\Longrightarrow A = 16\pi \left(\frac{GM}{c^2}\right)^2$

If you insist in speaking about a "radius", you may define the black hole's **areal radius** R by setting $A =: 4\pi R^2$

$\Longrightarrow$ for a Schwarzschild black hole: $R := \sqrt{\frac{A}{4\pi}} = \frac{2GM}{c^2} \simeq 3 \left(\frac{M}{M_\odot}\right) \ \mathrm{km}$

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Equilibrium in spacetime

A black hole *"in equilibrium"* is modeled by a stationary spacetime.

Definition of stationarity

A spacetime $(\mathscr{M}, \boldsymbol{g})$ is **stationary** iff it is invariant under the action of the translation group $(\mathbb{R}, +)$ and the orbits of the group action are timelike in the vicinity of the conformal infinity $\mathscr{I}$.

Killing vector

Let G be a 1-dimensional Lie group acting on $\mathscr{M}$ and $\boldsymbol{\xi}$ the vector field generating G.

$$G \text{ symmetry group of } (\mathscr{M}, \boldsymbol{g}) \iff \boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}}\, \boldsymbol{g} = 0 \text{ (Lie derivative along } \boldsymbol{\xi}\text{)}$$

$$\iff \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$$

$$\iff \exists \text{ coordinates } (t, x^1, x^2, x^3) \text{ such that } \frac{\partial g_{\alpha\beta}}{\partial t} = 0$$

The vector field $\boldsymbol{\xi}$ is then called a **Killing vector** of $(\mathscr{M}, \boldsymbol{g})$.

Equilibrium in spacetime

Equivalent definition of stationarity

A spacetime $(\mathscr{M}, \boldsymbol{g})$ is **stationary** iff there exists a Killing vector $\boldsymbol{\xi}$ that is timelike in the vicinity of the conformal infinity $\mathscr{I}$.

$\boldsymbol{\xi}$ is uniquely determined by requiring that it is future-directed near $\mathscr{I}$ and

$$\boldsymbol{\xi} \cdot \boldsymbol{\xi} \to -1 \quad \text{near } \mathscr{I}$$

$\Longrightarrow$ in asymptotically inertial coordinates (t, x, y, z), $\boldsymbol{\xi} = \dfrac{\partial}{\partial t}$

Killing horizons

Definition

A **Killing horizon** is a connected null hypersurface $\mathscr{H}$ in a spacetime $(\mathscr{M}, \boldsymbol{g})$ endowed with a Killing vector $\boldsymbol{\xi}$ such that, on $\mathscr{H}$, $\boldsymbol{\xi}$ is normal to $\mathscr{H}$:

$$\boldsymbol{\xi} \stackrel{\mathscr{H}}{=} \boldsymbol{\ell},$$

where $\boldsymbol{\ell}$ is a null normal to $\mathscr{H}$.

$\Longrightarrow$ $\boldsymbol{\xi}|_{\mathscr{H}} \neq 0$ and $\boldsymbol{\xi}|_{\mathscr{H}}$ is a null vector

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What about the event horizon $\mathscr{H}$ of a stationary black hole?

$\mathscr{H}$ is a null hypersurface that is stable (globally invariant) by the stationarity group action

$\Longrightarrow$ stationary Killing vector $\boldsymbol{\xi}$ is tangent to $\mathscr{H}$

A priori, this does not imply that $\mathscr{H}$ is a Killing horizon...

The event horizon as a Killing horizon

Rigidity theorem (Hawking 1972)

Let $(\mathscr{M}, \boldsymbol{g})$ be a stationary spacetime containing a black hole. Let $\mathscr{H}$ be a connected component of the event horizon. The stationary Killing vector $\boldsymbol{\xi}$ is either (i) null on all $\mathscr{H}$ or (ii) spacelike on some part of $\mathscr{H}$. In case (i), $\mathscr{H}$ is a Killing horizon w.r.t. $\boldsymbol{\xi}$. In case (ii), assume further that

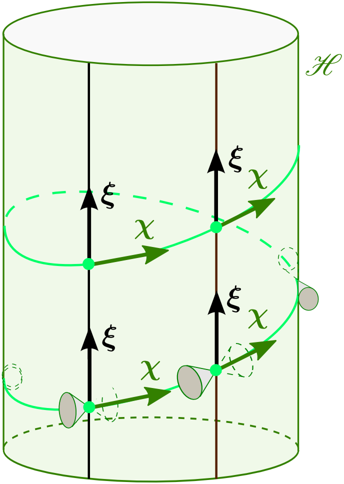
- $\mathscr{M}$ and $\mathscr{H}$ are (real) analytic manifolds and $\boldsymbol{g}$ is an analytic field,
- $\boldsymbol{g}$ fulfills the electrovacuum Einstein equation,
- $\mathscr{H}$ has compact cross-sections and $\boldsymbol{\xi}$ is transverse to them.

Then $(\mathscr{M}, \boldsymbol{g})$ admits a second Killing vector $\boldsymbol{\eta}$, generating a $\mathrm{SO}(2)$ action (axisymmetry) and there exists a constant $\Omega_{\mathscr{H}}$ such that $\mathscr{H}$ is a Killing horizon w.r.t. $\boldsymbol{\chi}$, where

$$\boldsymbol{\chi} := \boldsymbol{\xi} + \Omega_{\mathscr{H}} \boldsymbol{\eta}.$$

$\Omega_{\mathscr{H}}$ is called the **black hole rotation velocity**.

The event horizon as a Killing horizon



- $\boldsymbol{\xi}$: Killing vector generating the stationary group action; $\boldsymbol{\xi}$ is spacelike on $\mathscr{H}$
- $\boldsymbol{\chi}$: Killing vector normal to $\mathscr{H}$; $\boldsymbol{\chi}$ is null on $\mathscr{H}$ and tangent to $\mathscr{H}$'s null geodesic generators

$$\boldsymbol{\chi} = \boldsymbol{\xi} + \Omega_{\mathscr{H}} \boldsymbol{\eta}$$

Example: Kerr black hole

$\boldsymbol{\xi} = \boldsymbol{\partial}_t$, $\boldsymbol{\eta} = \boldsymbol{\partial}_\varphi$

$$\Omega_{\mathscr{H}} = \frac{J}{2M(M^2 + \sqrt{M^4 - J^2})}$$

Schwarzschild ($J = 0$): $\boldsymbol{\chi} = \boldsymbol{\xi}$ and $\Omega_{\mathscr{H}} = 0$

Surface gravity of a Killing horizon

Definition

Let $\mathscr{H}$ be a Killing horizon w.r.t. a Killing vector $\boldsymbol{\chi}$. The non-affinity coefficient κ of $\boldsymbol{\chi}$ considered as a null normal to $\mathscr{H}$, i.e. the coefficient κ such that

$$\boldsymbol{\nabla}_{\boldsymbol{\chi}}\boldsymbol{\chi} \stackrel{\mathscr{H}}{=} \kappa\,\boldsymbol{\chi},$$

is called the **surface gravity of $\mathscr{H}$**.

Example: Kerr black hole

$$\kappa = \frac{\sqrt{M^4 - J^2}}{2M(M^2 + \sqrt{M^4 - J^2})}$$

Schwarzschild limit $(J = 0)$: $\kappa = \dfrac{1}{4M}$

A surface gravity?

The genuine surface gravity of a black hole is the acceleration a felt by a corotating observer $\mathscr{O}$ just above the horizon in order not to fall into the black hole. It diverges when $\mathscr{O}$ is set closer and closer to $\mathscr{H}$:

$$\lim_{\mathscr{O}\to\mathscr{H}} a = +\infty$$

The finite quantity κ is actually a rescaled surface gravity:

$$\kappa = \lim_{\mathscr{O}\to\mathscr{H}} V a,$$

where V is the redshift factor of $\mathscr{O}$ with respect to a remote observer: $V = \sqrt{-\boldsymbol{\chi}\cdot\boldsymbol{\chi}} \to 0$ as $\mathscr{O} \to \mathscr{H}$.
For a Schwarzschild black hole: $V = (1 - r_{\mathscr{H}}/r)^{1/2}$.

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Physical interpretation of κ

For a static black hole, κ is the magnitude of the force exerted by a static observer at infinity to hold in place a particle of unit mass close to $\mathscr{H}$ by means of an infinitely long massless string.

Zeroth law of black hole dynamics

Constancy of surface gravity (Hawking, Carter 1973, Kay & Wald 1991)

Let $\mathscr{H}$ be a Killing horizon on $(\mathscr{M}, \boldsymbol{g})$. If

(1) $\boldsymbol{g}$ obeys Einstein's equation, with $\boldsymbol{T}$ fulfilling the *null dominant energy condition*: $-T^{\alpha}{}_{\mu}\ell^{\mu}$ is zero or future-directed causal vector for any future-directed null vector $\boldsymbol{\ell}$

or

(2) $\mathscr{H}$ is part of a *bifurcate Killing horizon*: $\boldsymbol{\chi} \to 0$ at some spacelike 2-surface bounding $\mathscr{H}$

or

(3) $(\mathscr{M}, \boldsymbol{g})$ is stationary, axisymmetric and invariant under $(t, \varphi) \mapsto (-t, -\varphi)$,

then the surface gravity κ is uniform over $\mathscr{H}$:

$$\boxed{\kappa = \text{const.}}$$

Remark: (1) requires general relativity, contrary to (2) and (3)

Zeroth law of black hole dynamics

When combined with the rigidity theorem (a stationary BH event horizon is a Killing horizon), the property $\kappa = \mathrm{const}$ leads to

Zeroth law of black hole dynamics

Under the hypotheses of the rigidity theorem and of the constancy of κ theorem, the surface gravity of the event horizon $\mathscr{H}$ of a stationary black hole is uniform over $\mathscr{H}$:

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$\Longrightarrow$ Analogy with (a consequence of) the zeroth law of thermodynamics: the temperature T of a body in equilibrium is uniform over the body

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Example: Kerr black hole

$$\kappa = \frac{\sqrt{M^4 - J^2}}{2M(M^2 + \sqrt{M^4 - J^2})} = \mathrm{const},$$

while a priori κ could have depended on the coordinate θ on $\mathscr{H}$.

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First law of black hole dynamics

First law of black hole dynamics (Bekenstein 1972, Bardeen, Carter & Hawking 1973)

In general relativity, the change δM in total mass between two nearby electrovacuum configurations of a black hole in equilibrium is related to the change δA in horizon area, the change δJ in angular momentum and the change δQ in electric charge by

$$\boxed{\delta M = \frac{\kappa}{8\pi}\,\delta A + \Omega_{\mathscr{H}}\,\delta J + \Phi_{\mathscr{H}}\,\delta Q}$$

$\Phi_{\mathscr{H}}$: horizon's electric potential (constant): $\Phi_{\mathscr{H}} \stackrel{\mathscr{H}}{=} -\boldsymbol{A}\cdot\boldsymbol{\chi} \stackrel{\mathscr{H}}{=} -A_\mu \chi^\mu$
($\boldsymbol{A}$: electromagnetic potential 1-form $\Longrightarrow$ electromagnetic field $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$)

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- δM $\sim$ energy variation (recall that $E = Mc^2$!)
- $\Omega_{\mathscr{H}}\,\delta J$ $\sim$ work performed by a torque on a body rotating at angular velocity $\Omega_{\mathscr{H}}$
- $\Phi_{\mathscr{H}}\,\delta Q$ $\sim$ work to change the electric charge of a body at electric potential $\Phi_{\mathscr{H}}$

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Null convergence/energy condition

$\boldsymbol{R}$ = Ricci tensor of metric $\boldsymbol{g}$ (part of the curvature tensor)

Null convergence condition

$$\boldsymbol{R}(\boldsymbol{\ell},\boldsymbol{\ell}) \geq 0 \quad \text{for any null vector } \boldsymbol{\ell}$$

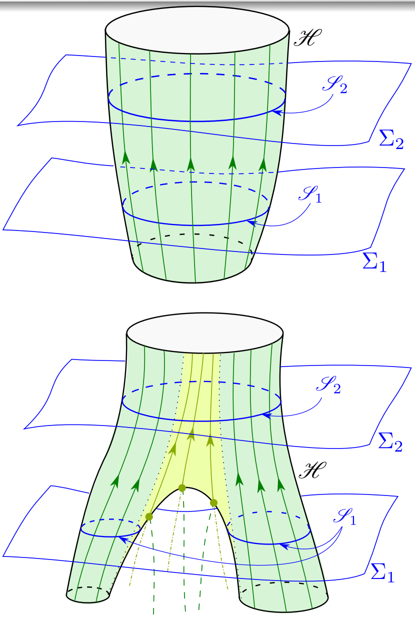
If gravitation is described by general relativity:
Einstein's equation $\Longrightarrow \boldsymbol{R}(\boldsymbol{\ell},\boldsymbol{\ell}) = 8\pi \boldsymbol{T}(\boldsymbol{\ell},\boldsymbol{\ell})$
$\boldsymbol{T}$ = energy-momentum tensor of matter and fields
$\Longrightarrow$ In general relativity, the null convergence condition is equivalent to

Null energy condition

$$\boldsymbol{T}(\boldsymbol{\ell},\boldsymbol{\ell}) \geq 0 \quad \text{for any null vector } \boldsymbol{\ell}$$

This is a very weak physical requirement: it is fulfilled by vacuum ($\boldsymbol{T} = 0$), standard matter ($\rho + p \geq 0$), any electromagnetic field and any massless scalar field.

Second law of black hole dynamics



Area theorem / Second law
(Hawking 1971, Chruściel, Delay, Galloway & Howard 2001)

Let $\mathscr{S}_1 = \mathscr{H} \cap \Sigma_1$ and $\mathscr{S}_2 = \mathscr{H} \cap \Sigma_2$, where Σ_1 and Σ_2 are two spacelike hypersurfaces, such that $\mathscr{S}_2$ lies in the causal future of $\mathscr{S}_1$: $\mathscr{S}_2 \subset J^+(\mathscr{S}_1)$.
If the null convergence condition is fulfilled and the black hole exterior is "well behaved" (globally hyperbolic), the area of $\mathscr{S}_2$ is greater than or equal to the area of $\mathscr{S}_1$:

$$A(\mathscr{S}_2) \geq A(\mathscr{S}_1)$$

Early history of the second law of black hole dynamics

- 1970: D. Christodoulou showed that in the process of particle accretion into a Kerr black hole, a certain function of M and J, the irreducible mass $M_{\rm irr}$, is always increasing, or stays constant in some idealized cases corresponding to reversible transformations. This contrasts with the black hole mass M, which may decrease (energy extraction by the Penrose process (1969)).

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- 1971: D. Christodoulou & R. Ruffini showed that $M_{\rm irr}^2 = A/(16\pi)$
- 1972: J. Bekenstein proposed to endow black holes with a genuine physical entropy $S = \eta k_{\rm B} \dfrac{A}{4\pi\hbar}$, with $\eta \sim 1$ by means of heuristic arguments

Entropy and temperature of a black hole?

Area theorem as thermodynamical second law $\Longrightarrow$ $\boxed{S = \alpha A}$
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Interpret the first law $\delta M = \dfrac{\kappa}{8\pi}\,\delta A + \underbrace{\Omega_{\mathscr{H}}\,\delta J + \Phi_{\mathscr{H}}\,\delta Q}_{\delta W}$

as a genuine thermodynamical first law $\delta E = T\,\delta S + \delta W$

$\Longrightarrow \dfrac{\kappa}{8\pi}\,\delta A = T\,\delta S$

$\Longrightarrow$ black hole temperature $\boxed{T = \dfrac{1}{8\pi\alpha}\kappa}$

Entropy and temperature of a black hole?

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$\Longrightarrow$ black hole temperature $\boxed{T = \frac{1}{8\pi\alpha}\kappa}$

$\Longrightarrow$ consistent with zeroth law: T uniform over a black hole in equilibrium

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What about the third law?

Planck-Nernst statement of the third law of thermodynamics

$S \to 0$ (or universal constant) as $T \to 0$.

$\Longrightarrow$ cannot hold for black holes since some of them (called *extremal*) have $\kappa = 0$ and $A \neq 0$.

Example: extremal Kerr black hole

$J = M^2 \Longrightarrow \kappa = 0$ and $A = 8\pi M^2$

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Unattainability statement of the third law of thermodynamics

It is impossible to bring any system to zero temperature by a finite number of operations.

$\Longrightarrow$ holds for astrophysical black holes accreting matter: can be spun up to $J = 0.998\, M^2$, not to the extremal state $J = M^2$ $(\kappa = 0)$ [Thorne (1974)]
$\Longrightarrow$ counterexamples (reaching the extremal Reissner-Nordström state) have been found recently [Kehle & Unger (2024)]

Shouldn't the temperature be absolute zero instead of $\propto \kappa$?

The laws of black hole dynamics, arising from classical (non-quantum) gravity, seem to open a nice path to black hole thermodynamics, but

> From the very nature of a black hole, its "true" thermodynamic temperature T must be zero.

Proof: consider a thermal reservoir of temperature $T_0 > 0$ in contact with the black hole. Energy flows from the reservoir to the black hole and not in the reverse way

$\Longrightarrow$ black hole temperature $T < T_0$, whatever T_0

$\Longrightarrow T = 0$ □

This contradicts the tentative identification $T = \dfrac{\kappa}{8\pi\alpha}$ ...

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However, we are going to see that

Taking into account quantum physics restores the identification $T \leftrightarrow \kappa$.

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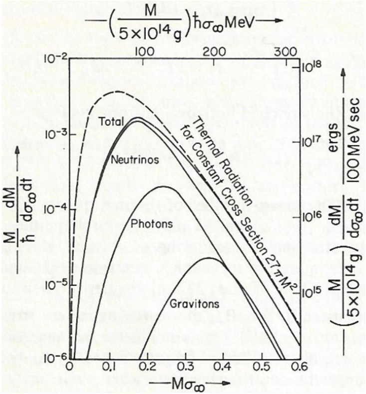
Quantum physics enters the game: Hawking radiation



Spectrum for a Schwarzschild black hole ($\kappa = (4M)^{-1}$) of mass $M \gg 10^{14}$ kg

$\sigma_\infty/(2\pi)$ = frequency at infinity

[Thorne, Zurek & Price (1986)]

Hawking radiation (Hawking 1974)

Let $(\mathscr{M}, \boldsymbol{g})$ be a stationary asymptotically flat spacetime that contains a black hole, the event horizon of which is a Killing horizon of constant surface gravity κ.

Quantum field theory in curved background predicts that any quantum field gives birth to a thermal radiation from the black hole to infinity, called **Hawking radiation**.

The radiation temperature as measured by *asymptotic* inertial observers, called the **Hawking temperature**, is

$$T_{\rm H} = \frac{\hbar}{k_{\rm B}} \frac{\kappa}{2\pi c}$$

Properties of Hawking radiation

- Hawking radiation is a prediction of semiclassical gravity (gravity is not quantized); it is independent of the metric theory of gravity (in particular, it does not rely on Einstein's equation).
- The Hawking temperature $T_{\rm H}$ is the temperature measured at infinity; a *static* observer $\mathscr{O}$ at finite distance measures $T = T_{\rm H}/V$, where V is the redshift factor: $V \to 0$ for $\mathscr{O} \to \mathscr{H}$, while a *free-falling* observer perceives no radiation at all ($T = 0$).
- For a Kerr black hole, $T_{\rm H} = \dfrac{\hbar}{k_{\rm B}} \dfrac{c^3}{8\pi G M} \dfrac{2}{1 + (1 - \bar{a}^2)^{-1/2}}$, where $\bar{a} := a/M = J/M^2$
- The Hawking temperature is tiny for astrophysical black holes:

$$T_{\rm H} = 6.17\ 10^{-8} \left(\frac{M_\odot}{M}\right) \frac{2}{1 + (1 - \bar{a}^2)^{-1/2}}\ {\rm K}$$

$\Longrightarrow$ Hawking radiation has not been detected, and will not be in the forseeable future (except for hypothetical micro black holes created in particle colliders)

Black hole evaporation

Backreaction to Hawking radiation $\Longrightarrow \dfrac{\mathrm{d}M}{\mathrm{d}t} = -\mathcal{A}\sigma T_{\mathrm{H}}^4 \sim -C\dfrac{\hbar}{M^2}$

$\Longrightarrow M(t) \sim \left(M_0^3 - 3C\hbar t\right)^{1/3}, \quad C \simeq 2.83\ 10^{-4}$

Evaporation time

A Schwarzschild black hole of initial mass M_0 fully evaporates via Hawking radiation within a time (measured at infinity)

$$t_{\mathrm{evap}} = \frac{M_0^3}{3C\hbar} = 1.54\ 10^{66}\left(\frac{M_0}{M_\odot}\right)^3\ \mathrm{yr}$$

For astrophysical BHs: $M_0 > 1M_\odot \Longrightarrow t_{\mathrm{evap}} > 10^{56}$ Universe age!

$t_{\mathrm{evap}} < t_{\mathrm{Univ}} \iff M_0 < 5.00\ 10^{11}\ \mathrm{kg}$ (mountain mass, proton-size BH)

Bekenstein-Hawking entropy

Recall: laws of BH dynamics $\Longrightarrow S = \alpha A$ and $T = \frac{1}{8\pi\alpha}\kappa$

Identifying T to the Hawking temperature $T_{\rm H}$ sets $\alpha = \frac{k_{\rm B}}{4\hbar}$

Hence $S = S_{\rm BH}$, with $\boxed{S_{\rm BH} = k_{\rm B}\frac{c^3 A}{4G\hbar}}$ **Bekenstein-Hawking entropy**

or, in terms of the Planck length $\ell_{\rm P} := \sqrt{\frac{G\hbar}{c^3}} \simeq 1.62\ 10^{-35}\ {\rm m}$,

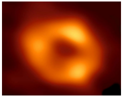
$$\boxed{S_{\rm BH} = k_{\rm B}\frac{A}{4\ell_{\rm P}^2}}$$

$\ell_{\rm P}$ tiny $\Longrightarrow S_{\rm BH}$ is huge!

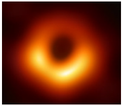
Black holes have a huge entropy!

For a Schwarzschild black hole:

$$A = 16\pi M^2 \Longrightarrow S_{\rm BH} = 4\pi k_{\rm B} \frac{GM^2}{\hbar c} = 1.05\ 10^{77} \left(\frac{M}{M_\odot}\right)^2 k_{\rm B}$$



Sgr A* $M \simeq 4\ 10^6\ M_\odot$ $S_{\rm BH} \simeq 2\ 10^{90}\ k_{\rm B}$



M87* $M \simeq 6\ 10^9\ M_\odot$ $S_{\rm BH} \simeq 4\ 10^{96}\ k_{\rm B}$

$\Longrightarrow$ compare with the total entropy of the observable Universe: $1.1\ 10^{90}\ k_{\rm B}$ (mostly from cosmic microwave and neutrino backgrounds, with only $\sim 10^{81}\ k_{\rm B}$ in all the stars)

The entropy of a single massive black hole, such as Sgr A* or M 87*, is larger than the entropy of the whole observable Universe!

The generalized second law (GSL)

In presence of black holes, the standard second law of thermodynamics has to be replaced by the

Generalized second law of thermodynamics (GSL) (Bekenstein 1973)

Define the **generalized entropy**

$$S_{\rm gen} := S_{\rm mat} + S_{\rm BH},$$

where $S_{\rm mat}$ is the ordinary entropy of matter and fields and $S_{\rm BH}$ is the Bekenstein-Hawking entropy of the black holes. Then, in any physical process, $S_{\rm gen}$ can only increase or stay constant:

$$\boxed{\Delta S_{\rm gen} \geq 0}$$

Status of the generalized second law (GSL)

$$\text{GSL:} \quad \Delta S_{\rm gen} = \Delta(S_{\rm mat} + S_{\rm BH}) \geq 0$$

Contrary to the second law of black hole *dynamics*, the GSL is not a theorem: it has not been proved.
Actually, as a thermodynamic statement, it can even be violated in some statiscally very unlikely processes. However:

The GSL has been checked in many processes involving black holes

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Example: Hawking evaporation

The GSL holds during the evaporation of a black hole:

Hawking radiation $\Longrightarrow \Delta M < 0 \Longrightarrow \Delta A < 0 \Longrightarrow \Delta S_{\rm BH} < 0$

but one can show that the entropy in the Hawking radiation is larger than $|\Delta S_{\rm BH}|$, so that $\Delta S_{\rm gen} > 0$.

Remark: since $\Delta A < 0$, the second law of BH dynamics (area theorem) is violated, but there is no issue since one of the hypotheses of the theorem is not fulfilled: the effective energy-momentum tensor of Hawking radiation does not obey the null energy condition.

Microscopic origin of the Bekenstein-Hawking entropy

GLS $\Longrightarrow$ $S_{\rm BH}$ same status as $S_{\rm mat}$: a genuine thermodynamic entropy!
$\Longrightarrow$ It should have some statistical meaning, i.e. count the number $\mathscr{N}$ of microstates for a given macroscopic black hole state, via Boltzmann's formula:

$$S_{\rm BH} = k_{\rm B} \ln \mathscr{N}$$

Black hole microstates? $\Longrightarrow$ quantum theory of gravity

Not existing yet, but two paths are explored:

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1. String theories

Bekenstein-Hawking formula $S_{\rm BH} = k_{\rm B} \dfrac{A}{4\ell_{\rm P}^2}$ recovered by Strominger & Vafa (1996) for an extremal ($\kappa = 0$) BH in some Yang-Mills theory in dimension 5.
Since then, $S_{\rm BH}$ recovered only for extremal or near-extremal BHs
No result for the Schwarzschild BH yet!

Microscopic origin of the Bekenstein-Hawking entropy

2. Loop quantum gravity (LQG)

Rovelli (1996), Ashtekar, Baez, Corichi & Krasnov (1997) have obtained

$$S_{\rm BH} = k_{\rm B} \frac{\gamma_0}{\gamma} \frac{A}{4\ell_{\rm P}^2}, \quad \gamma_0 \simeq 0.274$$

where γ = Barbero-Immirzi parameter of LGQ

γ determines the quantum of area as $a_0 = 4\sqrt{3}\pi\gamma\ell_{\rm P}^2$ but is not set by LQG

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Black hole entropy and the holographic principle

Striking feature of the black hole entropy: it is proportional to the area of the body, not to its "volume".
Thanks to the GSL, this feature extends to an upper bound for the entropy of any physical system:

Entropy bound

The entropy S of any system enclosed within a surface of area A must obey

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This entropy bound is at the origin of a cornerstone of contemporary theoretical physics:

Holographic principle ('t Hooft 1993, Susskind 1995)

Physics in a given spatial region can be fully described in terms of a set of degrees of freedom which reside on the surface bounding the region.

Recent developments

The laws of black hole dynamics have been extended in two directions:

1. **For general relativity in dimension $n > 4$:**
   All laws generalizes rather straightforwardly
   - $n = 5$ relevant for holographic approaches such as **gauge/gravity duality** (e.g. AdS/CFT)
   - no black hole uniqueness theorem for $n > 4$
2. **Beyond general relativity:**
   - zeroth law: actually does not depend on the gravity theory
   - first law: extension to any diffeomorphism-invariant gravity theory $\Longrightarrow$ Bekenstein-Hawking entropy (area) $\rightarrow$ **Wald entropy** [Wald (1993)] and generalizations [Dong (2014)], [Wall (2015)]
   - second law: work in progress [Hollands, Wald & Zhang (2024)] [Visser & Yan (2024)]

Conclusions

- The laws of black hole evolution derived from (classical) general relativity involve geometrical quantities, like A and κ, and display a striking analogy with the laws of thermodynamics, especially the irreversible evolution of the area (second law).
- Quantum field theory in curved spacetime promotes this analogy to a physically meaningful identification: the area *is* the entropy and the second law of thermodynamics becomes the GSL.
- Black hole thermodynamics has deep connections with any (tentative) quantum theory of gravitation; in particular any such theory must provide a microscopic explanation of the Bekenstein-Hawking formula for the entropy.

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