Figures of lecture 2

Geometry of null hypersurfaces and Killing horizons

Éric Gourgoulhon

Laboratoire d'étude de l'Univers et des phénomènes eXtrêmes (LUX)
Observatoire de Paris - PSL, CNRS, Sorbonne Université
Meudon, France

https://relativite.obspm.fr/blackholes/paris25/

PSL graduate program in Physics / Doctoral School PIF
ENS, Paris, France
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Home page for the lectures

https://relativite.obspm.fr/blackholes/paris25/

includes

- these slides
- the lecture notes (draft)
- some SageMath notebooks

Prerequisite

An introductory course on general relativity

Hypersurfaces in spacetime

A **hypersurface** of the n-dimensional spacetime $(\mathscr{M}, \boldsymbol{g})$ is an embedded submanifold of $\mathscr{M}$ of dimension $n-1$ (codimension 1).

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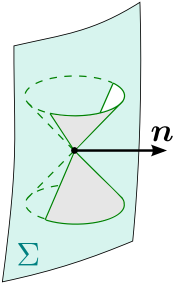
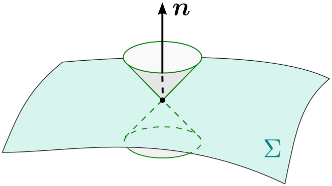
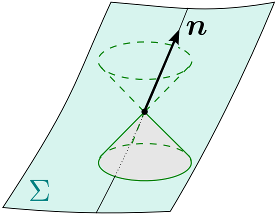
Locally, a hypersurface Σ can be of one of 3 types ($\boldsymbol{n}$ = normal to Σ):



Σ timelike
$\boldsymbol{g}|_\Sigma$ Lorentzian
$\boldsymbol{n}$ spacelike

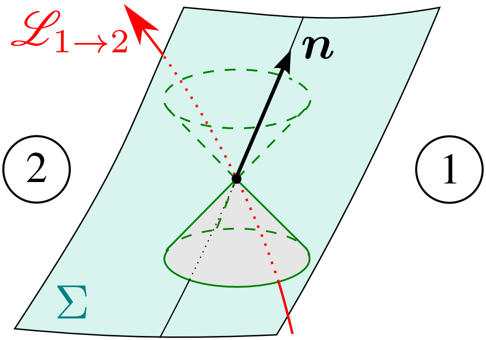


Σ spacelike
$\boldsymbol{g}|_\Sigma$ Riemannian
$\boldsymbol{n}$ timelike



Σ null
$\boldsymbol{g}|_\Sigma$ degenerate
$\boldsymbol{n}$ null (and tangent to Σ)

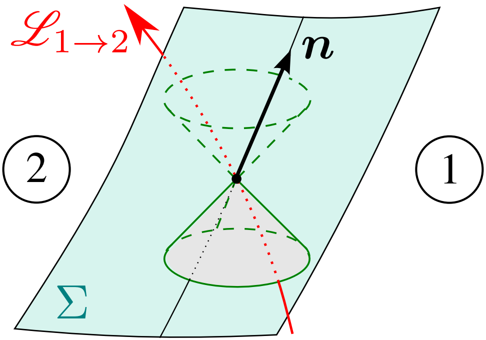
Null hypersurface as a causal boundary



For timelike worldlines $\mathscr{L}$ directed towards the future:

null hypersurface = 1-way membrane
$\Longrightarrow$ eligible for a black hole boundary...

Null hypersurface as a causal boundary



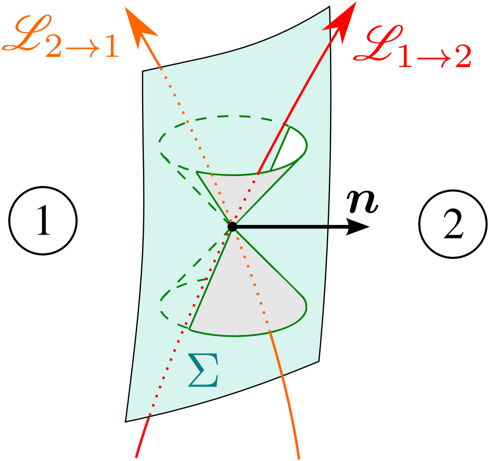
For timelike worldlines $\mathscr{L}$ directed towards the future:

null hypersurface = 1-way membrane
$\Longrightarrow$ eligible for a black hole boundary...
...and elected! (as a consequence of the formal definition of a black hole)

Theorem (Penrose 1968)

Wherever it is smooth, the event horizon of a black hole is a null hypersurface.

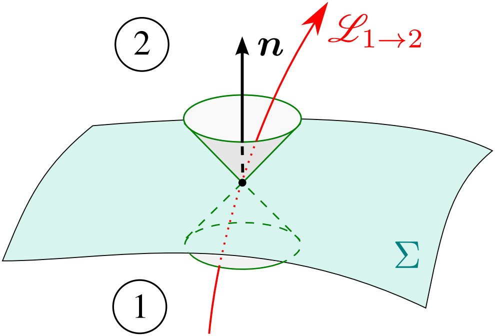
Timelike hypersurfaces are not causal boundaries



For timelike worldlines $\mathscr{L}$ directed towards the future:

timelike hypersurface = 2-way membrane
$\Longrightarrow$ not eligible for a black hole boundary

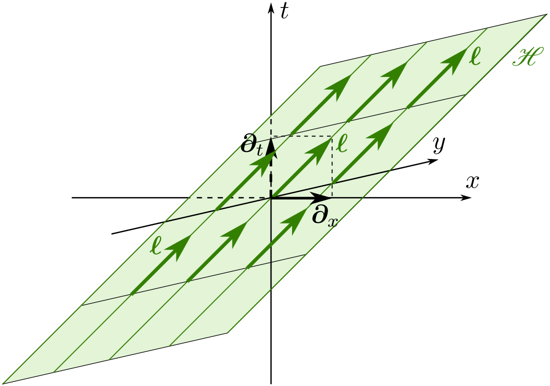
Spacelike hypersurfaces



For timelike worldlines $\mathscr{L}$ directed towards the future:

spacelike hypersurface = 1-way membrane
$\Longrightarrow$ in the dynamical black hole context: trapping horizons = spacelike hypersurfaces

Example 1: null hyperplane in Minkowski spacetime



$$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$$

$$\boxed{u := t - x = 0}$$

$$\mathbf{d}u = \mathbf{d}t - \mathbf{d}x$$

$$(\mathbf{d}u)_\alpha = \nabla_\alpha u = (1, -1, 0, 0)$$

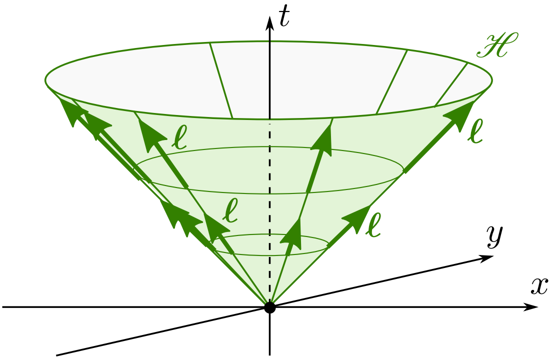
$$\nabla^\alpha u = (-1, -1, 0, 0)$$

Choose $\rho = 0$

$$\Longrightarrow \ell^\alpha = (1, 1, 0, 0)$$

$$\boxed{\boldsymbol{\ell} = \boldsymbol{\partial}_t + \boldsymbol{\partial}_x}$$

Example 2: future null cone in Minkowski spacetime



$$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$$

$$\boxed{u := t - \sqrt{x^2 + y^2 + z^2} = 0}$$

$$\mathbf{d}u = \mathbf{d}t - \frac{x}{r}\mathbf{d}x - \frac{y}{r}\mathbf{d}y - \frac{z}{r}\mathbf{d}z$$

$$r := \sqrt{x^2 + y^2 + z^2}$$

$$\nabla_\alpha u = \left(1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right)$$

$$\nabla^\alpha u = \left(-1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right)$$

Choose $\rho = 0$

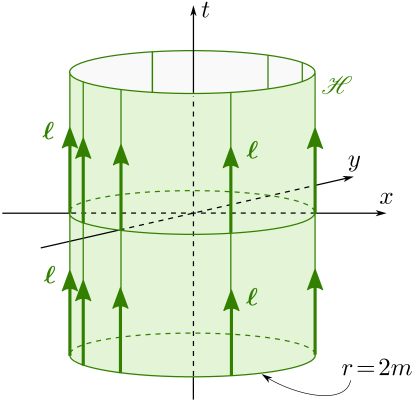
$$\Longrightarrow \ell^\alpha = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$$

$$\boxed{\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{x}{r}\boldsymbol{\partial}_x + \frac{y}{r}\boldsymbol{\partial}_y + \frac{z}{r}\boldsymbol{\partial}_z}$$

Example 3: Schwarzschild horizon

in Eddington-Finkelstein coordinates

$$\boldsymbol{g} = -\left(1 - \frac{2m}{r}\right) \mathbf{d}t^2 + \frac{4m}{r}\, \mathbf{d}t\, \mathbf{d}r + \left(1 + \frac{2m}{r}\right) \mathbf{d}r^2 + r^2 \mathbf{d}\theta^2 + r^2 \sin^2\theta\, \mathbf{d}\varphi^2$$



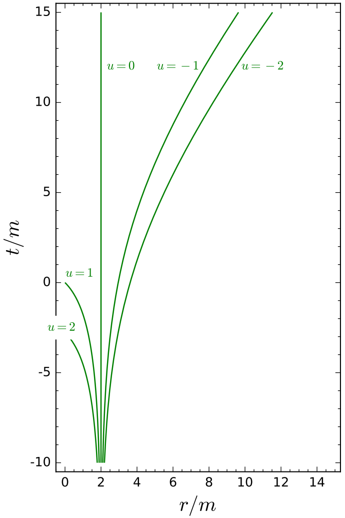
$$u := \left(1 - \frac{r}{2m}\right) \exp\left(\frac{r-t}{4m}\right) = 0$$

$$\mathscr{H} : \quad u = 0 \iff r = 2m$$

$$\mathbf{d}u = \frac{1}{4m} \mathrm{e}^{(r-t)/(4m)} \left[-\left(1 - \frac{r}{2m}\right) \mathbf{d}t - \left(1 + \frac{r}{2m}\right) \mathbf{d}r \right]$$

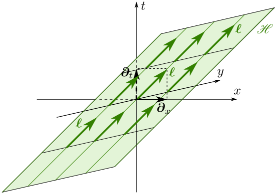
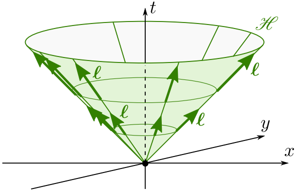
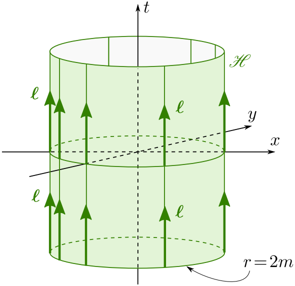
Exercise: compute $\boldsymbol{\ell}$ with ρ chosen so that $\ell^t = 1$ and get

$$\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{r - 2m}{r + 2m} \boldsymbol{\partial}_r \implies \boldsymbol{\ell} \overset{\mathscr{H}}{=} \boldsymbol{\partial}_t$$

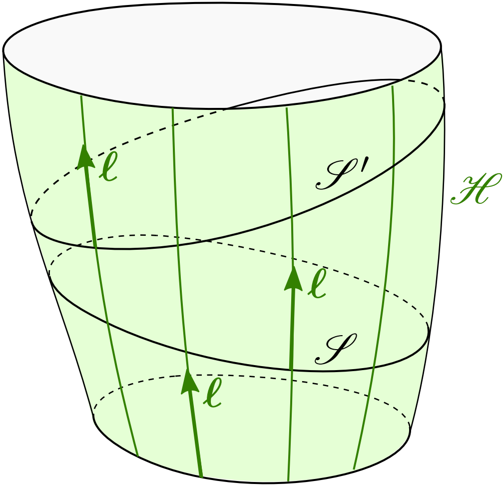


Hypersurfaces of constant value of u around the Schwarzschild horizon $u = 0$

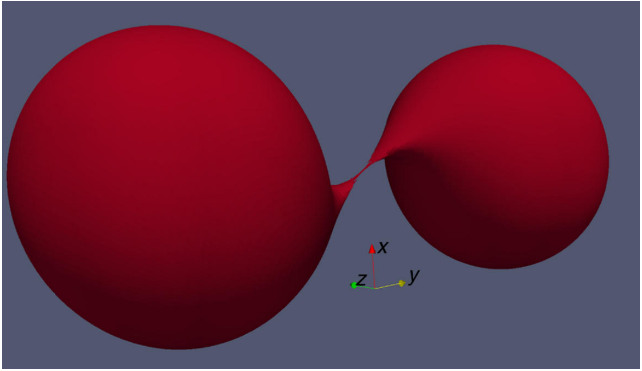
Examples of null geodesic generators

null hyperplane	future null cone	Schwarzschild horizon
		
$\nabla_{\boldsymbol{\ell}} \boldsymbol{\ell} = 0$	$\nabla_{\boldsymbol{\ell}} \boldsymbol{\ell} = 0$	$\nabla_{\boldsymbol{\ell}} \boldsymbol{\ell} = \kappa \boldsymbol{\ell}$
$\kappa = 0$	$\kappa = 0$	$\kappa = \frac{1}{4m}$

Cross-sections of a null hypersurface



Cross-section of the event horizon of a binary black hole merger

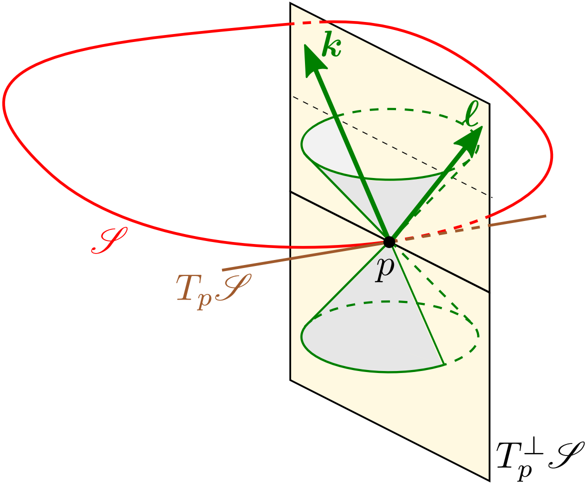


[Cohen, Kaplan & Scheel, PRD **85**, 024031 (2012)]

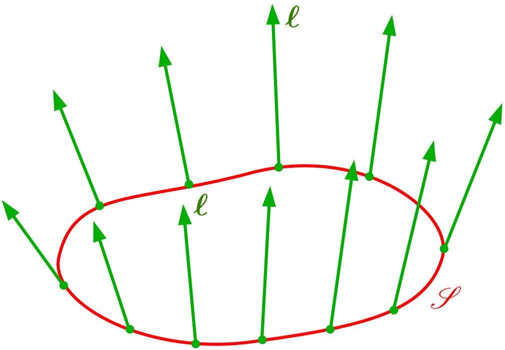
$\leftarrow$ First connected cross-section of the event horizon of an inspiralling binary black hole merger (slicing by coordinate time t)

(x, y)-*axes:* orbital plane

Orthogonal complement of a cross-section tangent space

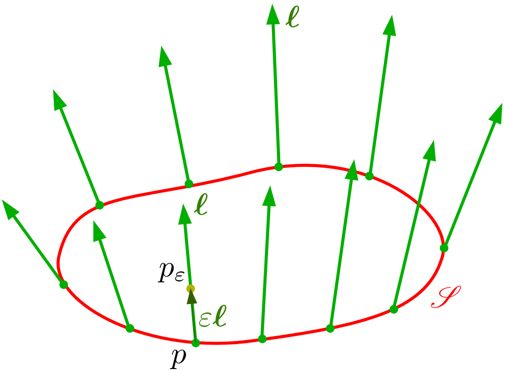


Expansion along a null normal



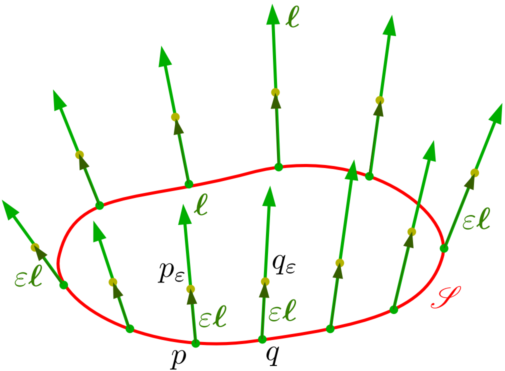
1. Consider a cross-section $\mathscr{S}$ and a null normal $\boldsymbol{\ell}$ to $\mathscr{H}$

Expansion along a null normal



1. Consider a cross-section $\mathscr{S}$ and a null normal $\boldsymbol{\ell}$ to $\mathscr{H}$
2. ε being a small parameter, displace the point p by the vector $\varepsilon\boldsymbol{\ell}$ to the point p_ε

Expansion along a null normal



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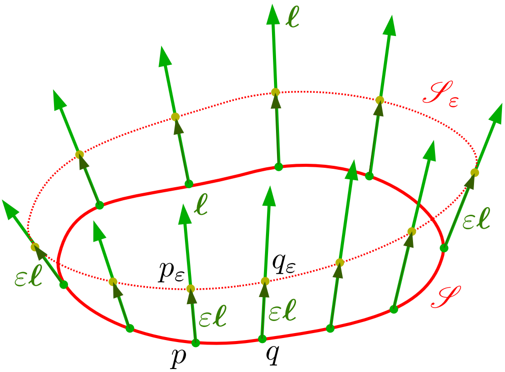
3. Do the same for each point in $\mathscr{S}$, keeping the value of ε fixed

Expansion along a null normal



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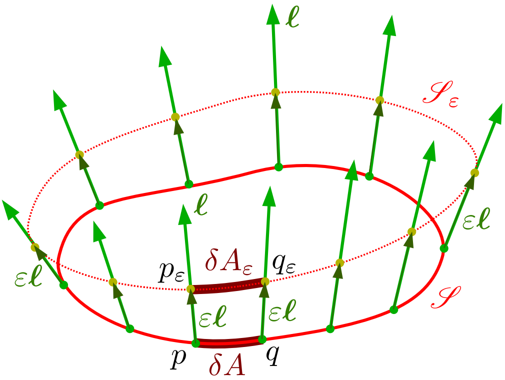

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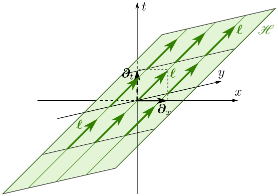
The **expansion along** $\boldsymbol{\ell}$ is defined from the relative change of the area δA (w.r.t. metric $\boldsymbol{q}$) of a surface element δS of $\mathscr{S}$ around p:

$$\theta_{(\boldsymbol{\ell})} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{\delta A_\varepsilon - \delta A}{\delta A} = \boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}} \ln \sqrt{q} = q^{\mu\nu} \nabla_\mu \ell_\nu$$

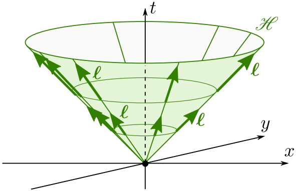
Examples of expansions

null hyperplane



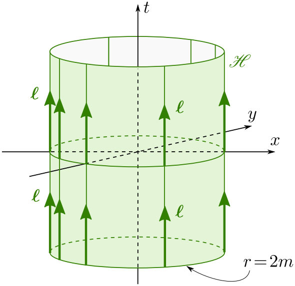
$$\theta_{(\boldsymbol{\ell})} = 0$$

future null cone



$$\theta_{(\boldsymbol{\ell})} = \frac{2}{r}$$

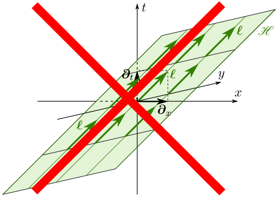
Schwarzschild horizon



$$\theta_{(\boldsymbol{\ell})} = 0$$

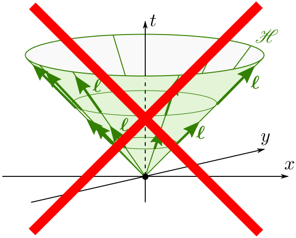
(Counter-)examples of non-expanding horizons

null hyperplane



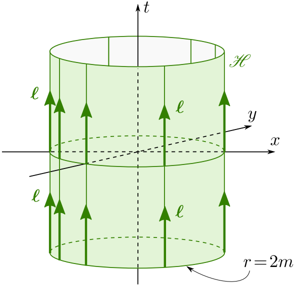
non-compact cross-sections

future null cone



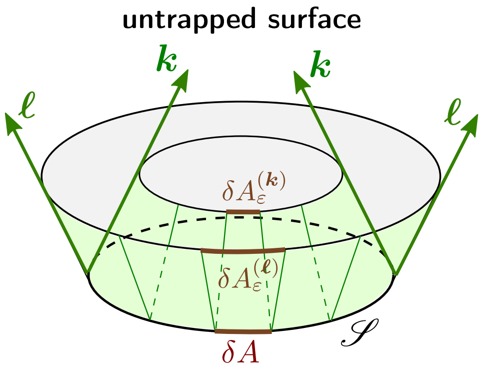
nonzero expansion

Schwarzschild horizon

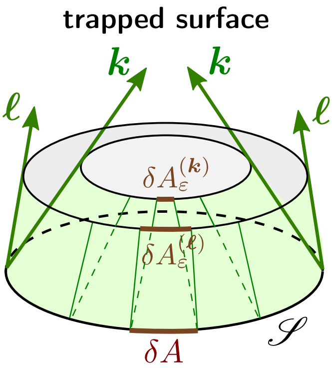


OK

Trapped surfaces



$\theta_{(\boldsymbol{k})} < 0$ and $\theta_{(\boldsymbol{\ell})} > 0$

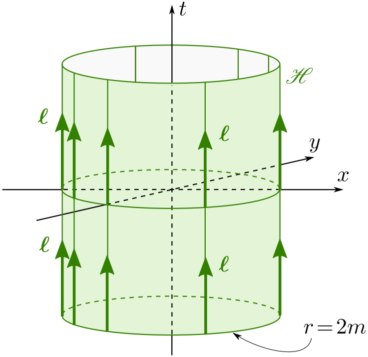


$\theta_{(\boldsymbol{k})} < 0$ and $\theta_{(\boldsymbol{\ell})} < 0$

Example: area of the Schwarzschild horizon

Spacetime metric:

$$\boldsymbol{g} = -\left(1 - \frac{2m}{r}\right)\mathbf{d}t^2 + \frac{4m}{r}\,\mathbf{d}t\,\mathbf{d}r + \left(1 + \frac{2m}{r}\right)\mathbf{d}r^2 + r^2\mathbf{d}\theta^2 + r^2\sin^2\theta\,\mathbf{d}\varphi^2$$



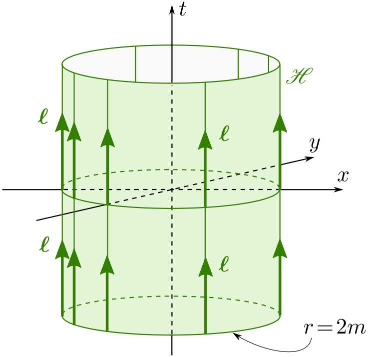
$\mathscr{H}$: $r = 2m$; coord: (t, θ, φ)

$\mathscr{S}$: $r = 2m$ and $t = t_0$; coord: $y^a = (\theta, \varphi)$

Example: area of the Schwarzschild horizon

Spacetime metric:

$$\boldsymbol{g} = -\left(1 - \frac{2m}{r}\right)\mathbf{d}t^2 + \frac{4m}{r}\,\mathbf{d}t\,\mathbf{d}r + \left(1 + \frac{2m}{r}\right)\mathbf{d}r^2 + r^2\mathbf{d}\theta^2 + r^2\sin^2\theta\,\mathbf{d}\varphi^2$$



$\mathscr{H}$: $r = 2m$; coord: (t, θ, φ)

$\mathscr{S}$: $r = 2m$ and $t = t_0$; coord: $y^a = (\theta, \varphi)$

$\Longrightarrow$ induced metric on $\mathscr{S}$:

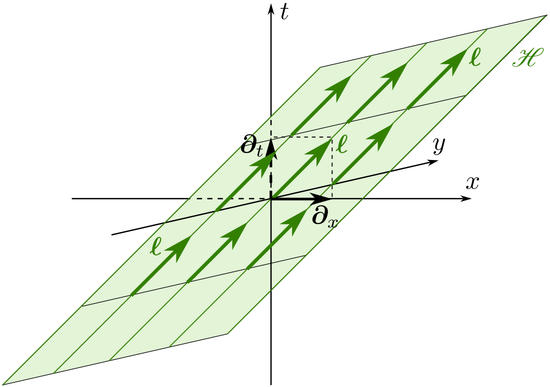
$$q_{ab}\,\mathrm{d}y^a\mathrm{d}y^b = (2m)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$$

$$\Longrightarrow q := \det(q_{ab}) = (2m)^4\sin^2\theta$$

$$\Longrightarrow A = \int_{\mathscr{S}} (2m)^2\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi$$

$$\Longrightarrow \boxed{A = 16\pi m^2}$$

Null hyperplane in Minkowski spacetime as a translation Killing horizon



$$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$$

$\mathscr{H}$: $u := t - x = 0$

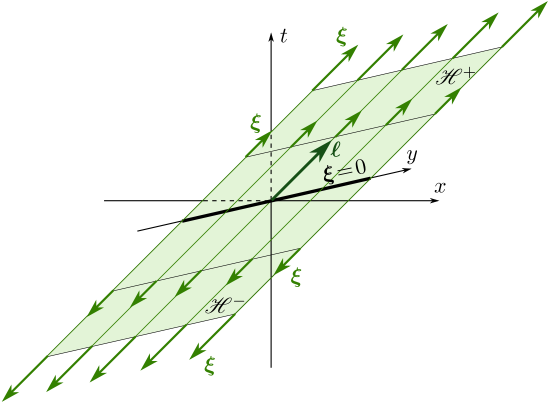
$\boldsymbol{\ell} = \boldsymbol{\partial}_t + \boldsymbol{\partial}_x$

$G = (\mathbb{R}, +)$ acting by translations in the direction $\boldsymbol{\partial}_t + \boldsymbol{\partial}_x$

Killing vector: $\boldsymbol{\xi} = \boldsymbol{\partial}_t + \boldsymbol{\partial}_x$

$\boldsymbol{\xi} \stackrel{\mathscr{H}}{=} \boldsymbol{\ell}$

Null half-hyperplane in Minkowski spacetime as a boost Killing horizon



$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$

$\mathscr{H}$: $u := t - x = 0$ and $t > 0$

null normal: $\boldsymbol{\ell} = \boldsymbol{\partial}_t + \boldsymbol{\partial}_x$

$G = (\mathbb{R}, +)$ acting by Lorentz boosts[a] in the (t, x) plane

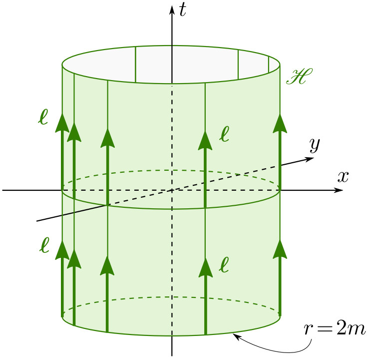
Killing vector: $\boldsymbol{\xi} = x\boldsymbol{\partial}_t + t\boldsymbol{\partial}_x$

$\boldsymbol{\xi} \overset{\mathscr{H}}{=} t(\boldsymbol{\partial}_t + \boldsymbol{\partial}_x) \overset{\mathscr{H}}{=} t\,\boldsymbol{\ell}$

[a] Parameter of G: boost rapidity

The Schwarzschild horizon as a Killing horizon

$$\boldsymbol{g} = -\left(1 - \frac{2m}{r}\right) \mathbf{d}t^2 + \frac{4m}{r}\,\mathbf{d}t\,\mathbf{d}r + \left(1 + \frac{2m}{r}\right)\mathbf{d}r^2 + r^2\mathbf{d}\theta^2 + r^2\sin^2\theta\,\mathbf{d}\varphi^2$$



$\mathscr{H}$: $r = 2m$

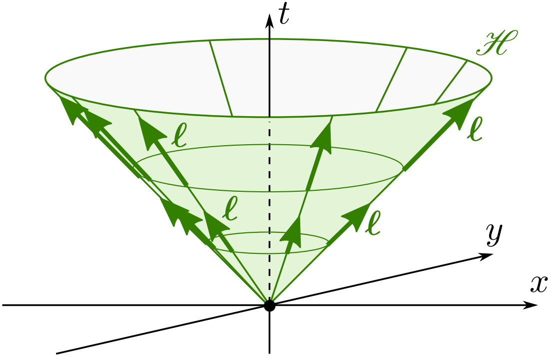
null normal: $\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{r-2m}{r+2m}\,\boldsymbol{\partial}_r$

$G = (\mathbb{R}, +)$ acting by (time) translation (stationarity)

Killing vector: $\boldsymbol{\xi} = \boldsymbol{\partial}_t$

$\boldsymbol{\xi} \overset{\mathscr{H}}{=} \boldsymbol{\ell}$

Counter-example: future null cone in Minkowski spacetime



$$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$$

$$\mathscr{H}: \ t - \sqrt{x^2 + y^2 + z^2} = 0$$

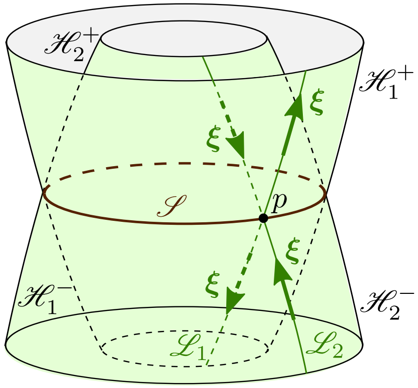
null normal:

$$\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{x}{r}\boldsymbol{\partial}_x + \frac{y}{r}\boldsymbol{\partial}_y + \frac{z}{r}\boldsymbol{\partial}_z$$

$\mathscr{H}$ is globally invariant under the action of the Lorentz group $\mathrm{O}(3,1)$, but its null generators are not invariant under the action of a single 1-dimensional subgroup of $\mathrm{O}(3,1)$.

Bifurcate Killing horizons

$(\mathscr{M}, \boldsymbol{g})$ = n-dimensional spacetime endowed with a Killing vector field $\boldsymbol{\xi}$



A **bifurcate Killing horizon** is the union

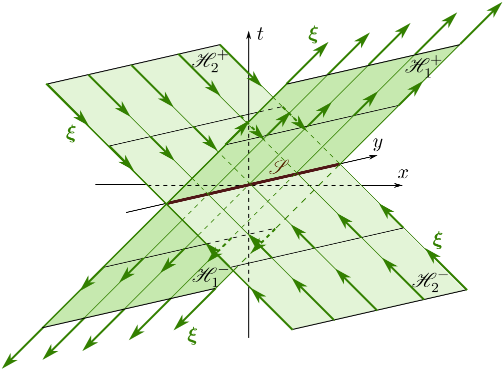
$$\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2,$$

where

- $\mathscr{H}_1$ and $\mathscr{H}_2$ are two null hypersurfaces;
- $\mathscr{S} := \mathscr{H}_1 \cap \mathscr{H}_2$ is a spacelike $(n-2)$-surface;
- each of the sets $\mathscr{H}_1 \setminus \mathscr{S}$ and $\mathscr{H}_2 \setminus \mathscr{S}$ has two connected components, which are Killing horizons w.r.t. $\boldsymbol{\xi}$.

The $(n-2)$-dimensional submanifold $\mathscr{S}$ is called the **bifurcation surface** of $\mathscr{H}$.

Example 1: bifurcate Killing horizon w.r.t. a Lorentz boost generator



$(\mathscr{M}, \boldsymbol{g})$: Minkowski spacetime

$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$

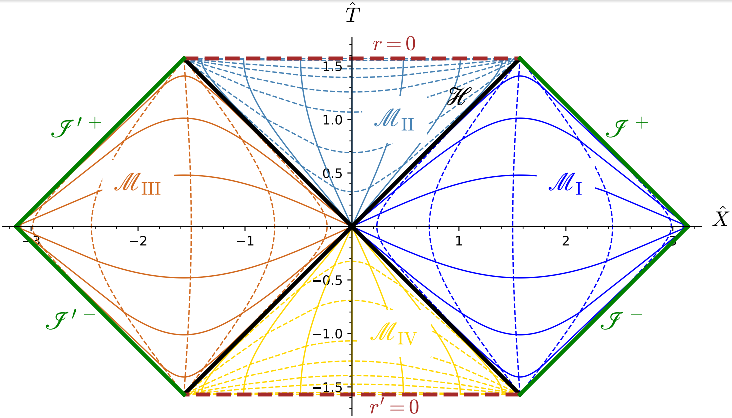
Killing vector: $\boldsymbol{\xi} = x\boldsymbol{\partial}_t + t\boldsymbol{\partial}_x$
$\Longrightarrow$ generates Lorentz boosts in the plane (t, x)

$\mathscr{H}_1$: $t = x$

$\mathscr{H}_2$: $t = -x$

$\mathscr{S}$: $(t, x) = (0, 0)$

Example 2: bifurcate Killing horizon in Schwarzschild spacetime

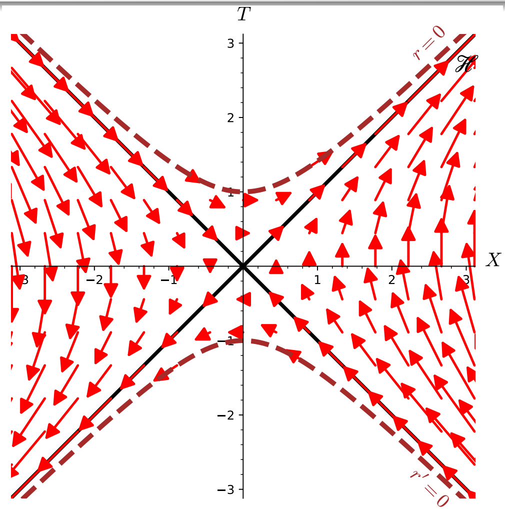


Dashed lines: field lines of the stationary Killing vector ξ

Thick black lines: bifurcate Killing horizon w.r.t. ξ

https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/Schwarz_conformal_std.ipynb

Example 2: bifurcate Killing horizon in Schwarzschild spacetime



Stationary Killing vector $\boldsymbol{\xi}$ in the maximal extension of Schwarzschild spacetime (Kruskal diagram)

https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/Schwarz_Kruskal_Szekeres.ipynb