Figures of lecture 1

Definition and main properties of black holes

Éric Gourgoulhon

Laboratoire d'étude de l'Univers et des phénomènes eXtrêmes (LUX)
Observatoire de Paris - PSL, CNRS, Sorbonne Université
Meudon, France

https://relativite.obspm.fr/blackholes/paris25/

PSL graduate program in Physics / Doctoral School PIF
ENS, Paris, France
30 April 2025

Home page for the lectures

https://relativite.obspm.fr/blackholes/paris25/

includes

- these slides
- the lecture notes (draft)
- some SageMath notebooks

Prerequisite

An introductory course on general relativity

Spacetime

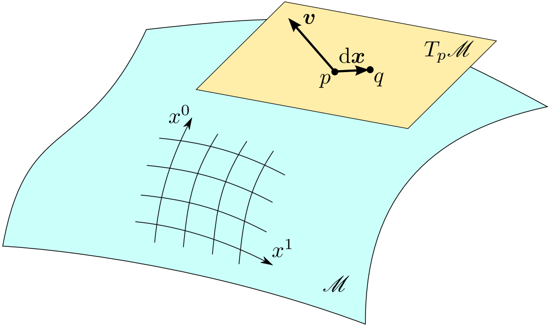
spacetime $= (\mathscr{M}, \boldsymbol{g})$

- $\mathscr{M}$: n-dimensional smooth manifold ($n \geq 3$)
- $\boldsymbol{g}$: Lorentzian metric on $\mathscr{M}$

Spacetime

spacetime = $(\mathscr{M}, \boldsymbol{g})$

- $\mathscr{M}$: n-dimensional smooth manifold ($n \geq 3$)
- $\boldsymbol{g}$: Lorentzian metric on $\mathscr{M}$

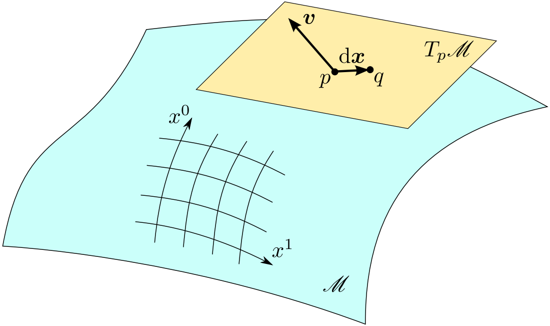


Smooth manifold: topological space $\mathscr{M}$ that *locally* resembles $\mathbb{R}^n$ (but maybe not globally)
$\Longrightarrow$ **coordinate charts**
$\Longrightarrow$ **tangent vectors**

Spacetime

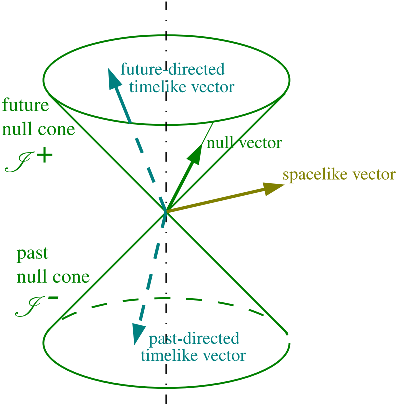
spacetime = $(\mathscr{M}, \boldsymbol{g})$

- $\mathscr{M}$: n-dimensional smooth manifold ($n \geq 3$)
- $\boldsymbol{g}$: Lorentzian metric on $\mathscr{M}$



Smooth manifold: topological space $\mathscr{M}$ that *locally* resembles $\mathbb{R}^n$ (but maybe not globally)
$\Longrightarrow$ **coordinate charts**
$\Longrightarrow$ **tangent vectors**

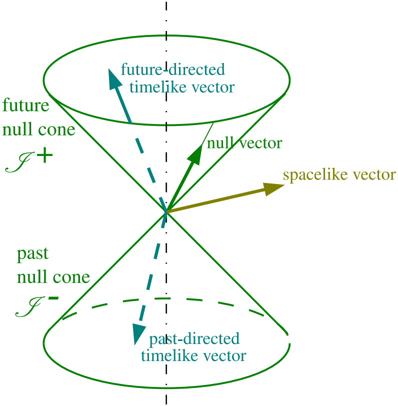
Remark: vector connecting two points p and q defined only for p and q infinitely close



Vector $\boldsymbol{v} \in T_p\mathscr{M}$ is

- **spacelike** $\iff$ $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) > 0$
- **null** $\iff$ $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = 0$
- **timelike** $\iff$ $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) < 0$

Metric's null cone



Vector $\boldsymbol{v} \in T_p\mathscr{M}$ is

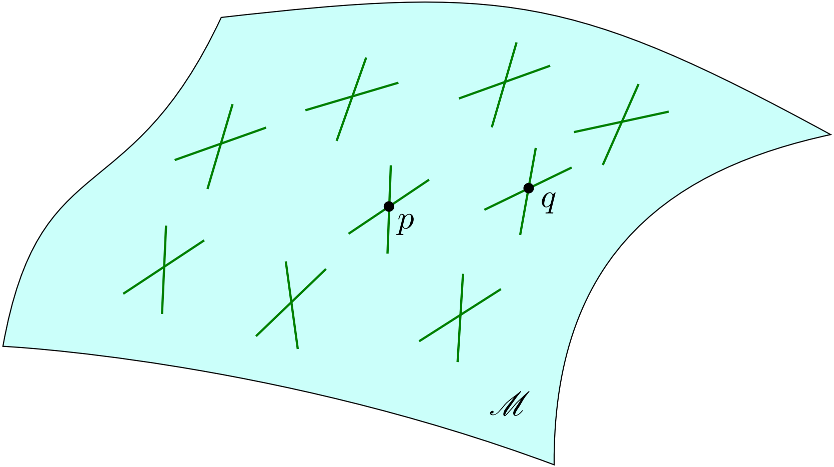
- **spacelike** $\iff \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) > 0$
- **null** $\iff \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = 0$
- **timelike** $\iff \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) < 0$

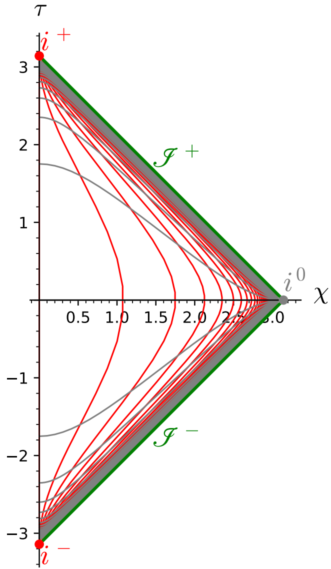
Additional assumption:

the spacetime $(\mathscr{M}, \boldsymbol{g})$ is **time-oriented**

$\Longrightarrow$ future and past directions

Lorentzian manifold $(\mathscr{M}, \boldsymbol{g})$



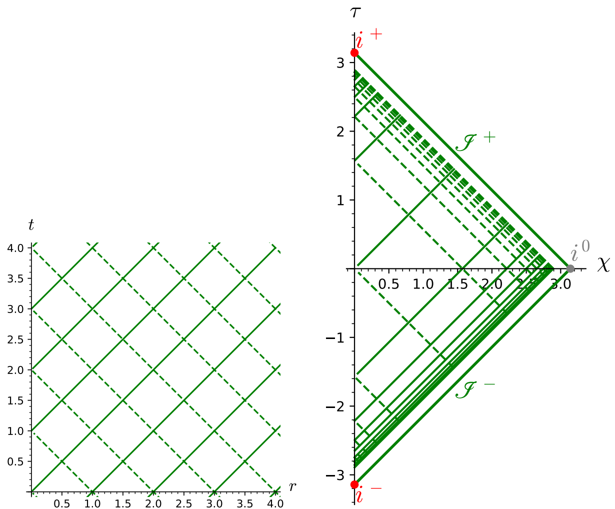


View in the (τ, χ) coordinate plane

$0 \leq \chi < \pi$
$\chi - \pi < \tau < \pi - \chi$

red: $r = \text{const}$
grey: $t = \text{const}$

Conformal completion of Minkowski spacetime
Conformal diagram



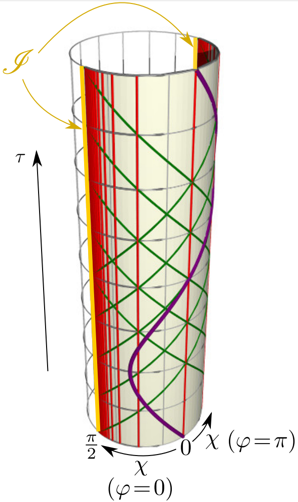
Radial null geodesics:
solid:
$u := t - r = \text{const}$
dashed:
$v := t + r = \text{const}$

Radial null geodesics appear as straight lines with $\pm 45°$ slope

$\Longrightarrow$ conformal diag. also called Penrose diagram or Carter-Penrose diagram

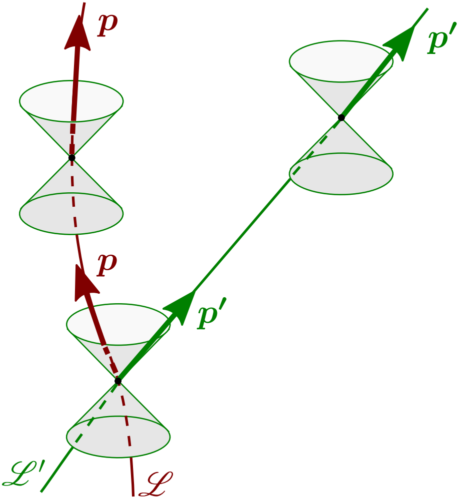
Conformal completion of anti-de Sitter spacetime

Embedding into the Einstein cylinder



- on $\mathscr{E}$:
  $-\infty < \tau < +\infty$
  $0 \leq \chi \leq \pi$
- on $\mathscr{M}$:
  $-\infty < \tau < +\infty$
  $0 \leq \chi < \pi/2$

cf. https://nbviewer.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_anti_de_Sitter.ipynb for an interactive 3D view



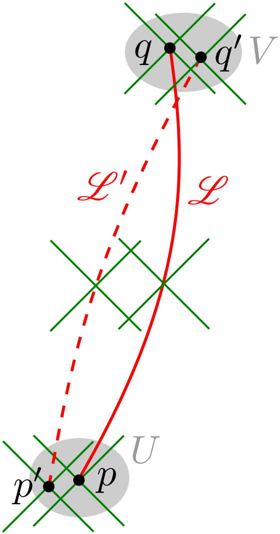
Particle described by its spacetime extent: **worldline** $\mathscr{L}$

massive part. $\iff$ timelike worldline
massless part. $\iff$ null worldline
(tachyon $\iff$ spacelike worldline)

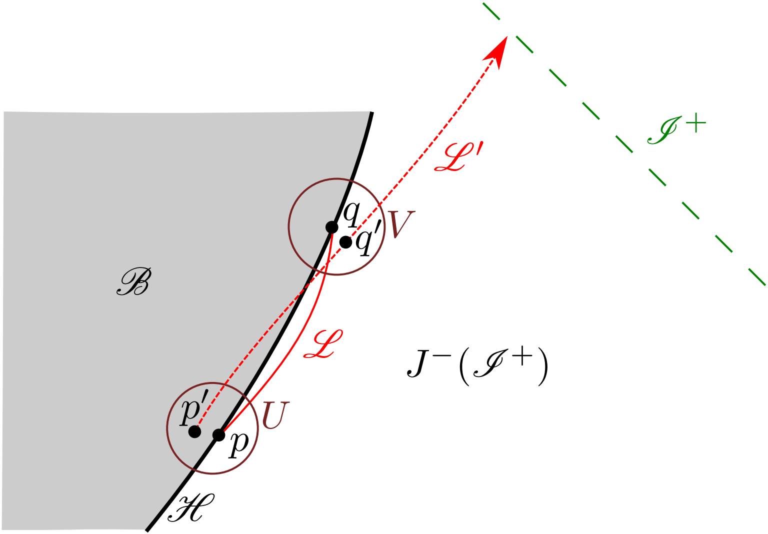
Dynamics of a *simple* particle (no spin, no internal structure) entirely described by a future-directed vector field tangent to the worldline: the **energy-momentum** $\boldsymbol{p}$

Particle's mass: $\boxed{m = \sqrt{-\boldsymbol{g}(\boldsymbol{p}, \boldsymbol{p})}}$

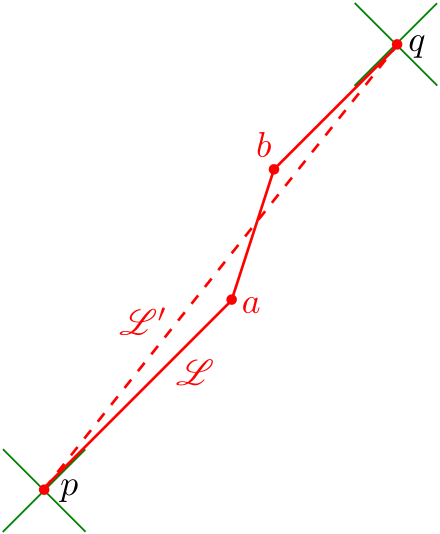
Lemma: stability of timelike curves



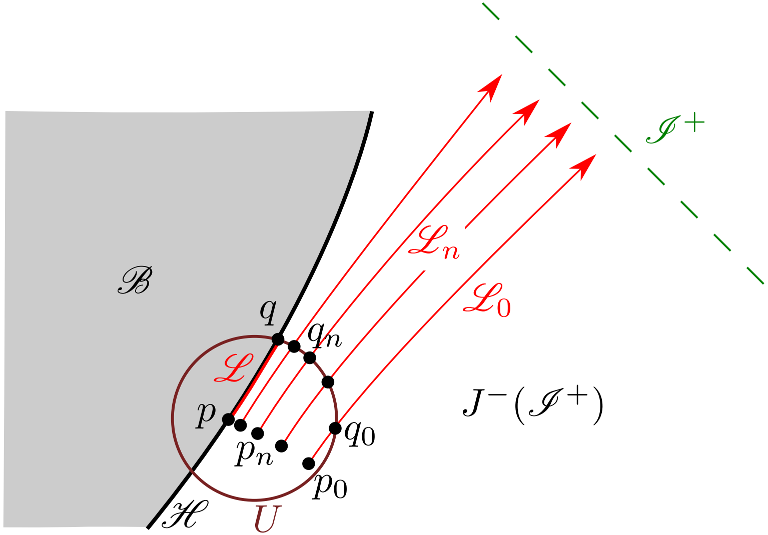
Proving that $\mathscr{H}$ is achronal



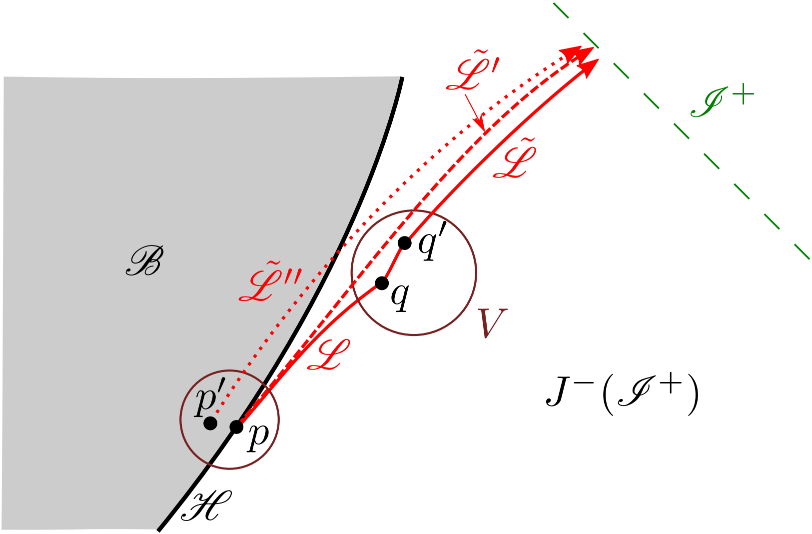
The timelike segment lemma



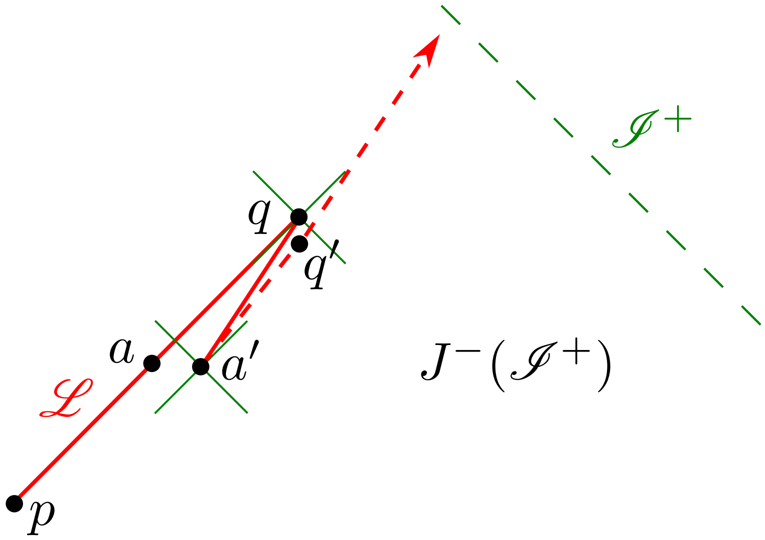
Causal curve connecting p to q



Proving by contradiction that q lies in $\mathscr{H}$

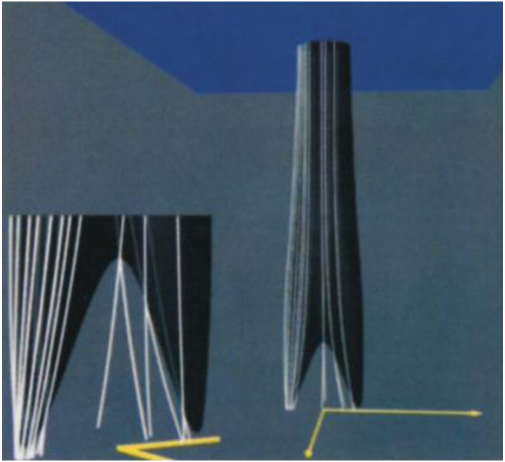


Proving that $\mathscr{L}$ lies entirely in $\mathscr{H}$



Event horizon of a binary black hole merger

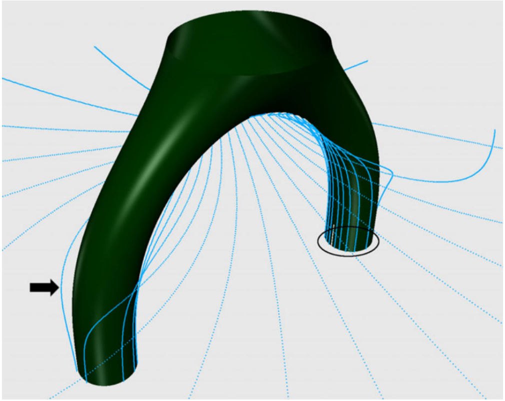
Head-on merger



[R.A. Matzner et al., Science **270**, 941 (1995)]

Event horizon of a binary black hole merger

Head-on merger



[Cohen, Pfeiffer & Scheel, CQG **26**, 035005 (2009)]

Movies of the head-on black hole merger

These movies correspond to the article

[Cohen, Pfeiffer & Scheel, CQG **26**, 035005 (2009)]

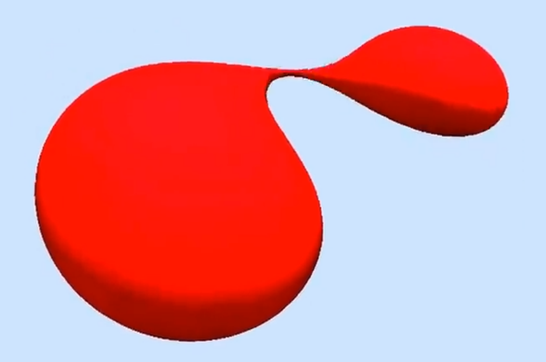
and are provided by courtesy of Harald Pfeiffer (AEI, Potsdam)

- Cross-sections of the event horizon
- Event horizon and its generators
- Event horizon with external geodesics becoming generators

Event horizon of an inspiral black hole merger

Merger of two spinning ($J/M^2 \simeq 0.4$) black holes of mass ratio 2:1 inspiralling from an initial circular orbit computed by Szilágyi, Lindblom & Scheel [PRD **80**, 124010 (2009)]

Movie of cross-sections of constant coordinate time of the event horizon:

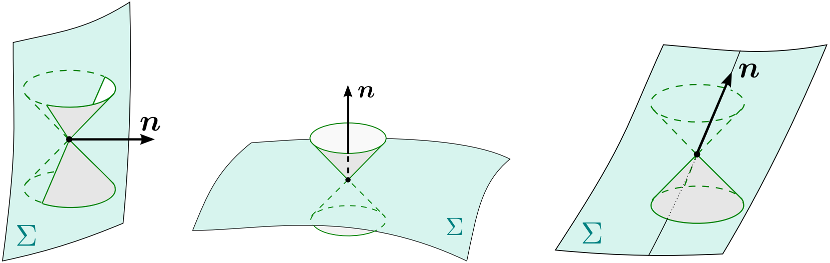


Source: https://www.black-holes.org/explore/movies

Three kinds of hypersurfaces

Boundary in spacetime $\Longrightarrow$ $(n-1)$-dimensional submanifold, i.e. **hypersurface**

Locally, a hypersurface Σ can be of one of 3 types:

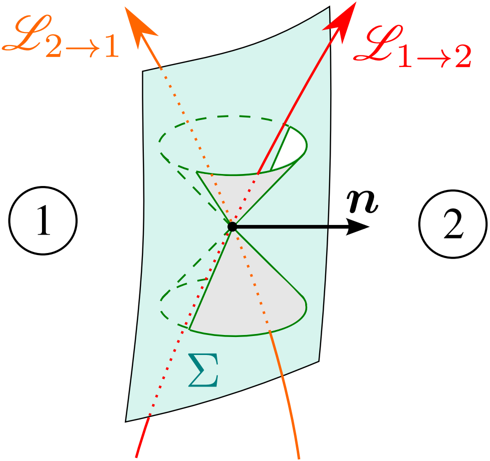


Σ timelike
$\boldsymbol{g}|_\Sigma$ Lorentzian
$\boldsymbol{n}$ spacelike

Σ spacelike
$\boldsymbol{g}|_\Sigma$ Riemannian
$\boldsymbol{n}$ timelike

Σ null
$\boldsymbol{g}|_\Sigma$ degenerate
$\boldsymbol{n}$ null (and tangent to Σ)

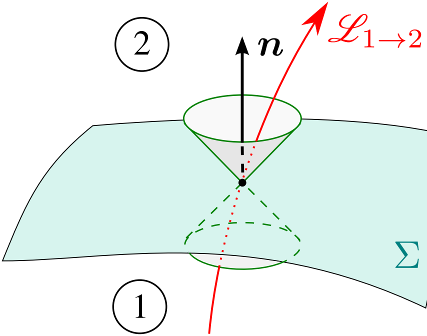
Timelike hypersurface



For worldlines $\mathscr{L}$ directed towards the future:

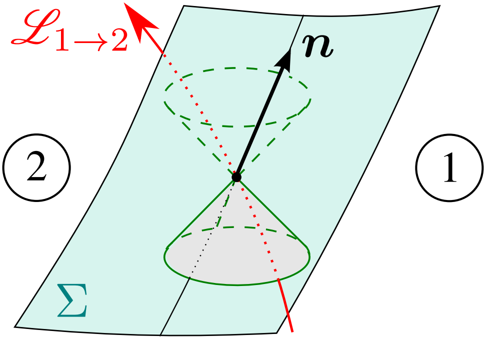
timelike hypersurface = 2-way membrane
$\Longrightarrow$ not eligible for a black hole boundary

Spacelike hypersurface



For worldlines $\mathscr{L}$ directed towards the future:

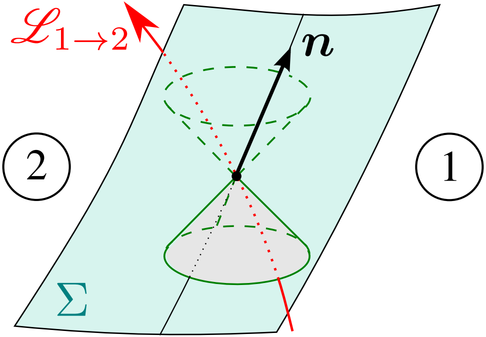
spacelike hypersurface =
1-way membrane
$\Longrightarrow$ eligible for a black hole boundary



For worldlines $\mathscr{L}$ directed towards the future:

null hypersurface = 1-way membrane
$\Longrightarrow$ eligible for a black hole boundary...

Null hypersurface



For worldlines $\mathscr{L}$ directed towards the future:

null hypersurface = 1-way membrane
$\Longrightarrow$ eligible for a black hole boundary...
...and elected! (as a consequence of the BH definition)

The event horizon of a black hole is a topological hypersurface of spacetime. Wherever it is smooth, it is a null hypersurface.