Figures of lecture 2

Geometry of null hypersurfaces and Killing horizons

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https://relativite.obspm.fr/blackholes/paris23/

PSL graduate programs in Physics and in Astrophysics
ENS, Paris, France
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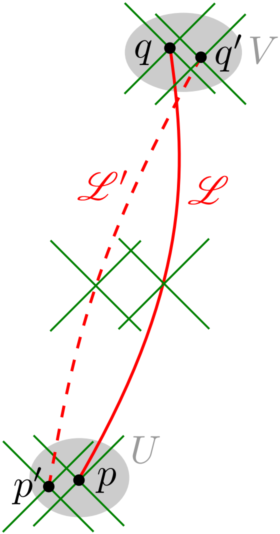
Home page for the lectures

https://relativite.obspm.fr/blackholes/paris23

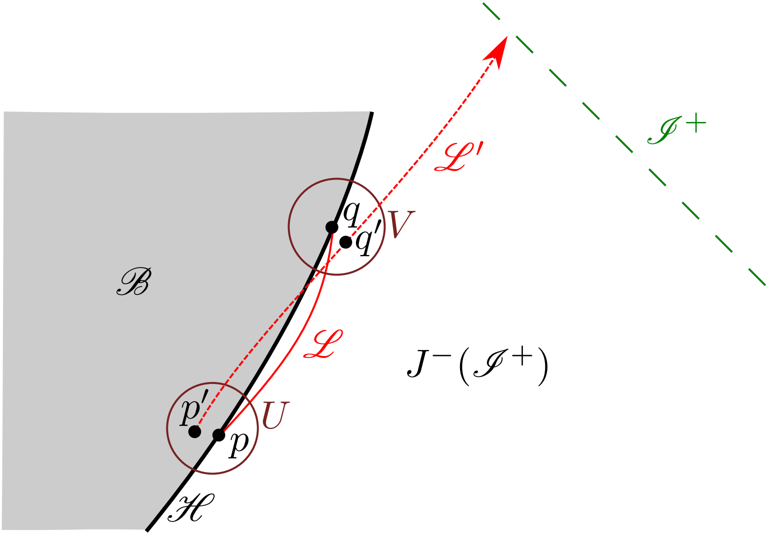
includes

- the lecture notes (draft)
- some SageMath notebooks
- these slides

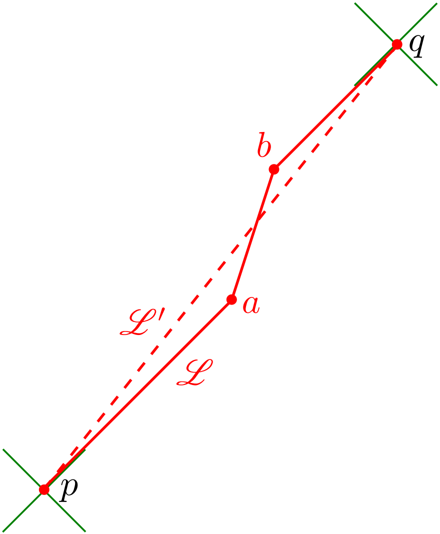
Lemma: stability of timelike curves



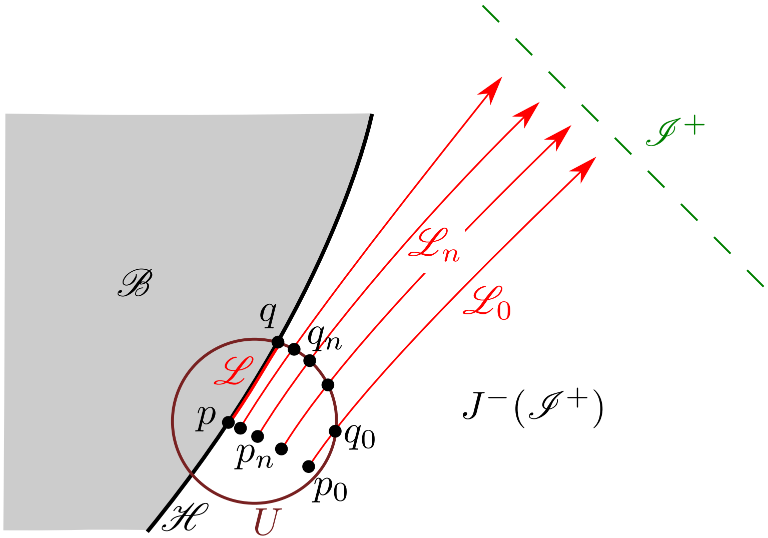
Proving that $\mathscr{H}$ is achronal



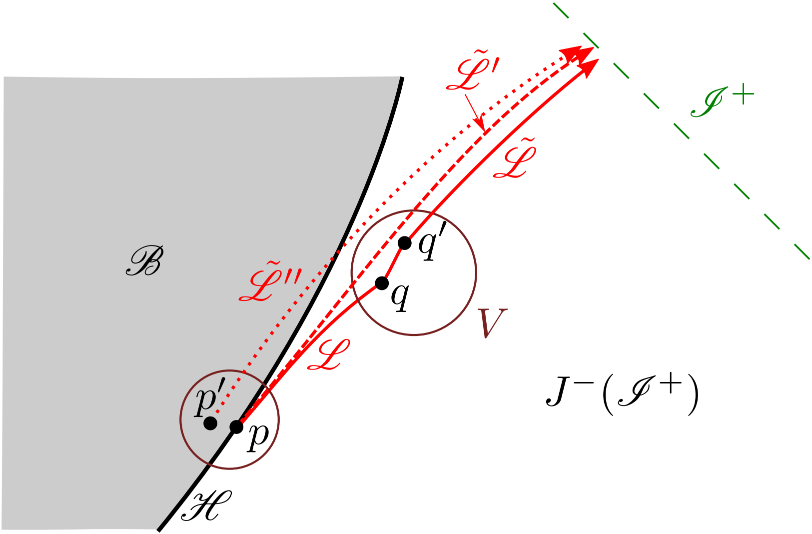
The timelike segment lemma

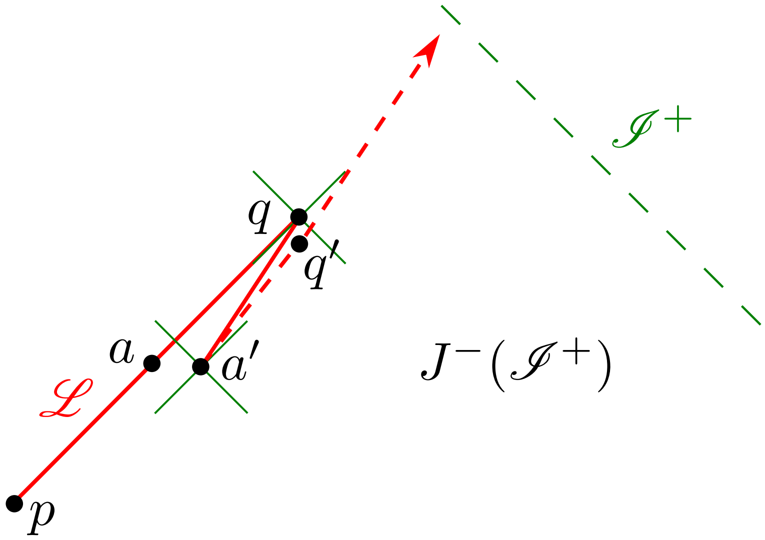


Causal curve connecting p to q



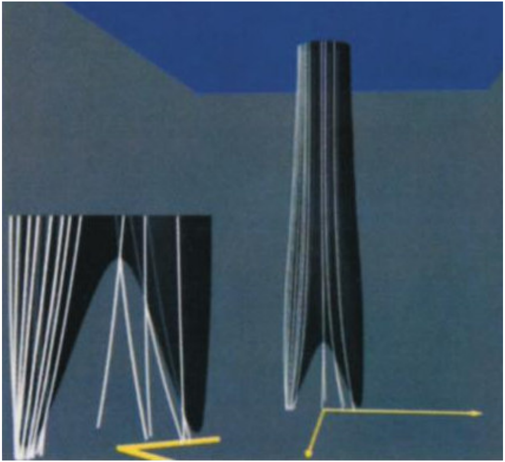
Proving by contradiction that q lies in $\mathscr{H}$




$\mathscr{I}^+$
q
q'
a
a'
$J^-(\mathscr{I}^+)$
$\mathscr{L}$
p

Event horizon of a binary black hole merger

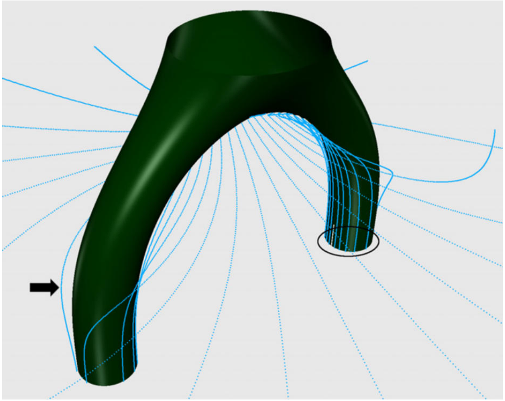
Head-on merger



[R.A. Matzner et al., Science **270**, 941 (1995)]

Event horizon of a binary black hole merger

Head-on merger

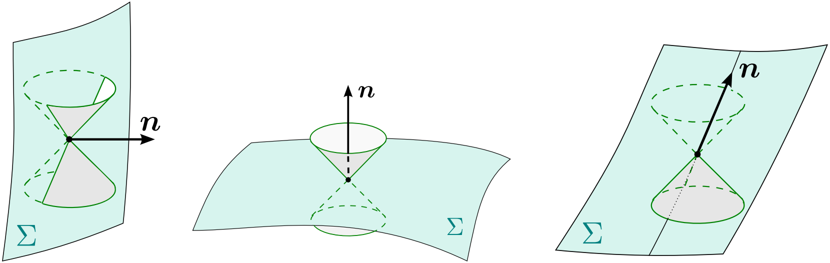


[Cohen, Pfeiffer & Scheel, CQG **26**, 035005 (2009)]

Three kinds of hypersurfaces

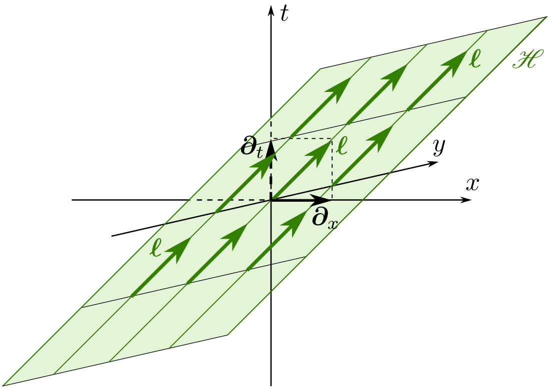
Boundary in spacetime $\Longrightarrow$ $(n-1)$-dimensional submanifold, i.e. **hypersurface**

Locally, a hypersurface Σ can be of one of 3 types:



Σ timelike	Σ spacelike	Σ null
$\boldsymbol{g}\|_\Sigma$ Lorentzian	$\boldsymbol{g}\|_\Sigma$ Riemannian	$\boldsymbol{g}\|_\Sigma$ degenerate
$\boldsymbol{n}$ spacelike	$\boldsymbol{n}$ timelike	$\boldsymbol{n}$ null (and tangent to Σ)

Example 1: null hyperplane in Minkowski spacetime



$$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$$

$$\boxed{u := t - x = 0}$$

$$\mathbf{d}u = \mathbf{d}t - \mathbf{d}x$$

$$(\mathbf{d}u)_\alpha = \nabla_\alpha u = (1, -1, 0, 0)$$

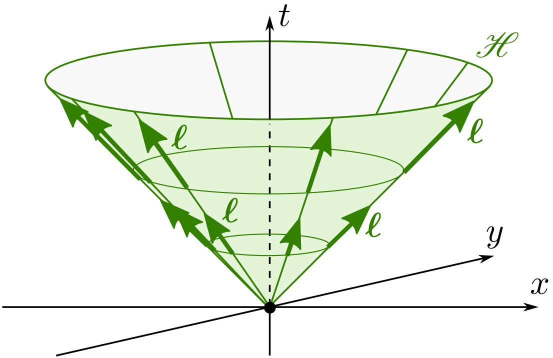
$$\nabla^\alpha u = (-1, -1, 0, 0)$$

Choose $\rho = 0$

$$\Longrightarrow \ell^\alpha = (1, 1, 0, 0)$$

$$\boxed{\boldsymbol{\ell} = \boldsymbol{\partial}_t + \boldsymbol{\partial}_x}$$

Example 2: future null cone in Minkowski spacetime



$$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$$

$$\boxed{u := t - \sqrt{x^2 + y^2 + z^2} = 0}$$

$$\mathbf{d}u = \mathbf{d}t - \frac{x}{r}\mathbf{d}x - \frac{y}{r}\mathbf{d}y - \frac{z}{r}\mathbf{d}z$$

$$r := \sqrt{x^2 + y^2 + z^2}$$

$$\nabla_\alpha u = \left(1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right)$$

$$\nabla^\alpha u = \left(-1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right)$$

Choose $\rho = 0$

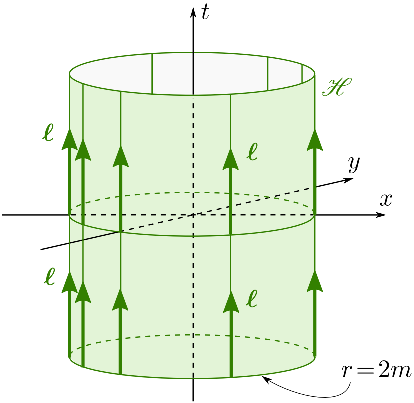
$$\Longrightarrow \ell^\alpha = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$$

$$\boxed{\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{x}{r}\boldsymbol{\partial}_x + \frac{y}{r}\boldsymbol{\partial}_y + \frac{z}{r}\boldsymbol{\partial}_z}$$

Example 3: Schwarzschild horizon

in Eddington-Finkelstein coordinates

$$\boldsymbol{g} = -\left(1 - \frac{2m}{r}\right) \mathbf{d}t^2 + \frac{4m}{r} \mathbf{d}t\, \mathbf{d}r + \left(1 + \frac{2m}{r}\right) \mathbf{d}r^2 + r^2 \mathbf{d}\theta^2 + r^2 \sin^2\theta\, \mathbf{d}\varphi^2$$



$$u := \left(1 - \frac{r}{2m}\right) \exp\left(\frac{r-t}{4m}\right) = 0$$

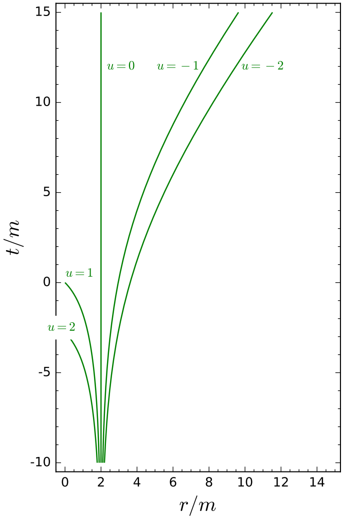
$$\mathscr{H} : \quad u = 0 \iff r = 2m$$

$$\mathbf{d}u = \frac{1}{4m} e^{(r-t)/(4m)} \left[-\left(1 - \frac{r}{2m}\right) \mathbf{d}t - \left(1 + \frac{r}{2m}\right) \mathbf{d}r \right]$$

Exercise: compute $\boldsymbol{\ell}$ with ρ chosen so that $\ell^t = 1$ and get

$$\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{r-2m}{r+2m} \boldsymbol{\partial}_r \implies \boldsymbol{\ell} \overset{\mathscr{H}}{=} \boldsymbol{\partial}_t$$

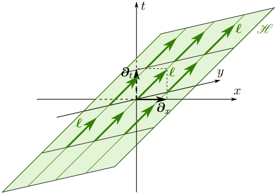
Example 3: Schwarzschild horizon
in Eddington-Finkelstein coordinates



Hypersurfaces of constant value of u around the Schwarzschild horizon $u = 0$

Examples of null geodesic generators

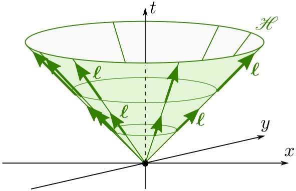
null hyperplane



$$\nabla_{\boldsymbol{\ell}}\, \boldsymbol{\ell} = 0$$

$$\kappa = 0$$

future null cone



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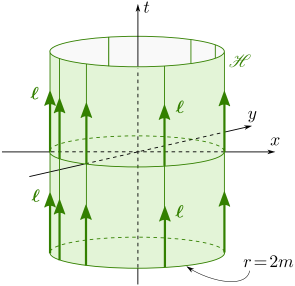
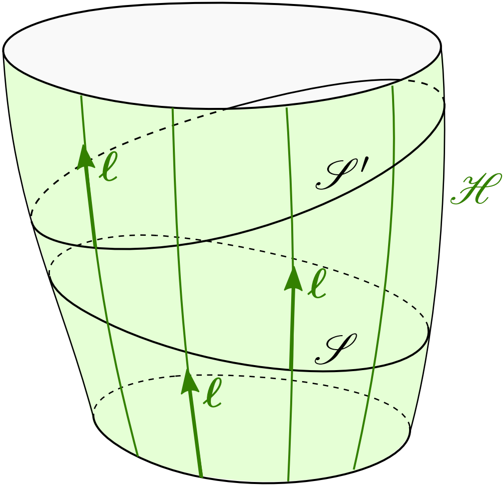
Schwarzschild horizon



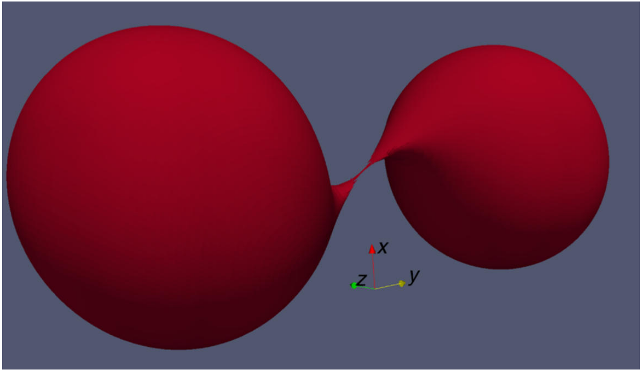
$$\nabla_{\boldsymbol{\ell}}\, \boldsymbol{\ell} = \kappa\, \boldsymbol{\ell}$$

$$\kappa = \frac{1}{4m}$$

Cross-sections of a null hypersurface



Cross-section of the event horizon of a binary black hole merger

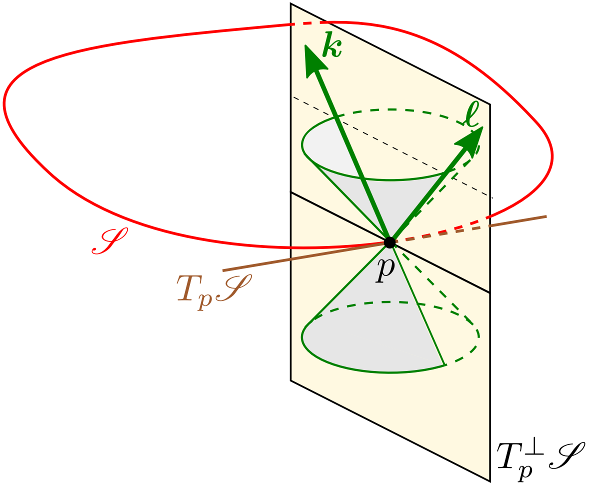


[Cohen, Kaplan & Scheel, PRD **85**, 024031 (2012)]

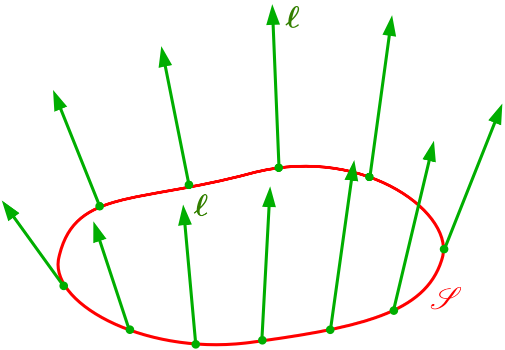
← First connected cross-section of the event horizon of an inspiralling binary black hole merger (slicing by coordinate time t)

(x, y)-*axes:* orbital plane

Orthogonal complement of a cross-section tangent space

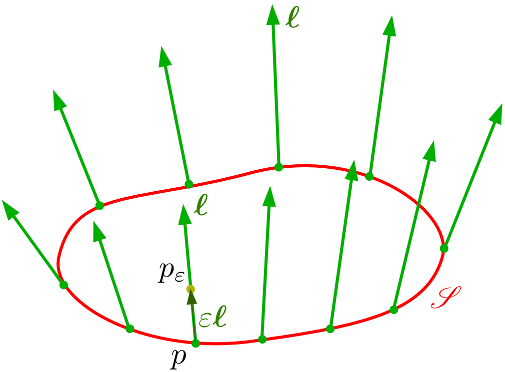


Expansion along a null normal



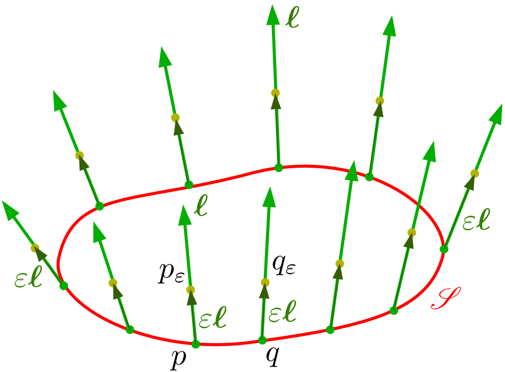
1. Consider a cross-section $\mathscr{S}$ and a null normal $\boldsymbol{\ell}$ to $\mathscr{H}$

Expansion along a null normal



1. Consider a cross-section $\mathscr{S}$ and a null normal ℓ to $\mathscr{H}$
2. ε being a small parameter, displace the point p by the vector $\varepsilon\ell$ to the point p_ε

Expansion along a null normal



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3. Do the same for each point in $\mathscr{S}$, keeping the value of ε fixed

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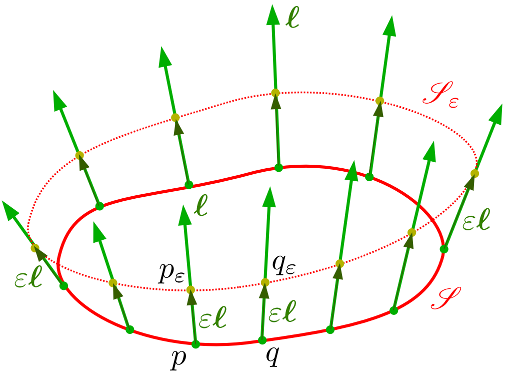

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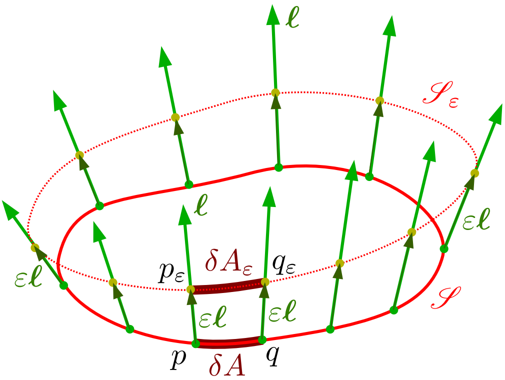
4. Since $\boldsymbol{\ell}$ is tangent to $\mathscr{H}$, this defines a new cross-section $\mathscr{S}_\varepsilon$



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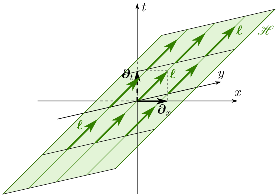
At each point, the **expansion along $\boldsymbol{\ell}$** is defined from the relative change of the area element δA:

$$\theta_{(\boldsymbol{\ell})} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{\delta A_\varepsilon - \delta A}{\delta A} = \mathcal{L}_{\boldsymbol{\ell}} \ln \sqrt{q} = q^{\mu\nu} \nabla_\mu \ell_\nu$$

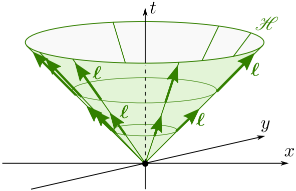
Examples of expansions

null hyperplane



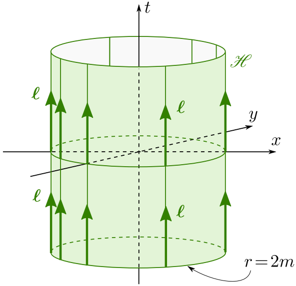
$$\theta_{(\boldsymbol{\ell})} = 0$$

future null cone



$$\theta_{(\boldsymbol{\ell})} = \frac{2}{r}$$

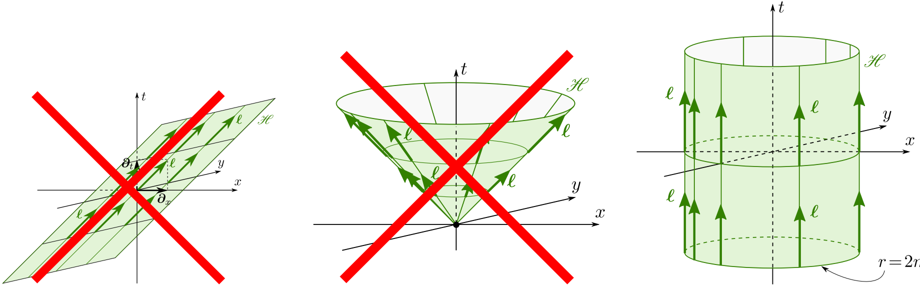
Schwarzschild horizon



$$\theta_{(\boldsymbol{\ell})} = 0$$

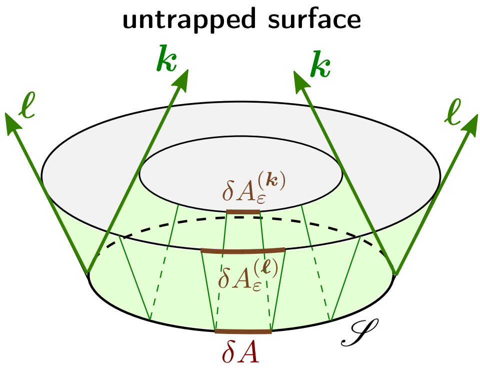
(Counter-)examples of non-expanding horizons

null hyperplane | future null cone | Schwarzschild horizon

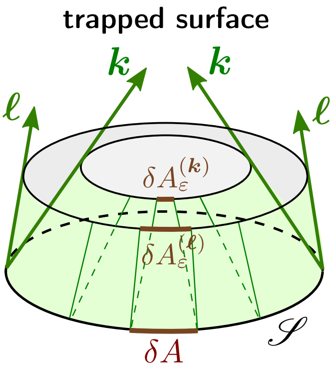


no closed cross-sections | nonzero expansion | OK

Trapped surfaces



$\theta_{(\boldsymbol{k})} < 0$ and $\theta_{(\boldsymbol{\ell})} > 0$

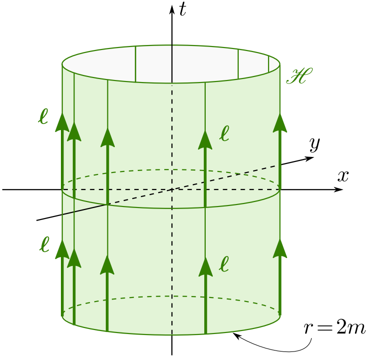


$\theta_{(\boldsymbol{k})} < 0$ and $\theta_{(\boldsymbol{\ell})} < 0$

Example: area of the Schwarzschild horizon

Spacetime metric:

$$\boldsymbol{g} = -\left(1 - \frac{2m}{r}\right) \mathbf{d}t^2 + \frac{4m}{r}\, \mathbf{d}t\, \mathbf{d}r + \left(1 + \frac{2m}{r}\right) \mathbf{d}r^2 + r^2 \mathbf{d}\theta^2 + r^2 \sin^2\theta\, \mathbf{d}\varphi^2$$



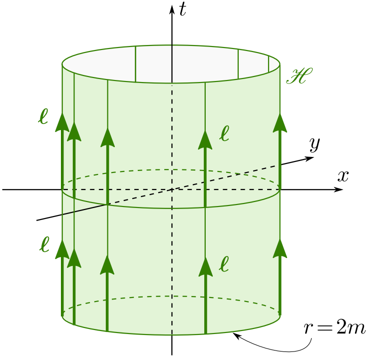
$\mathscr{H}$: $r = 2m$; coord: (t, θ, φ)

$\mathscr{S}$: $r = 2m$ and $t = t_0$; coord: (θ, φ)

Example: area of the Schwarzschild horizon

Spacetime metric:

$$\boldsymbol{g} = -\left(1 - \frac{2m}{r}\right)\mathbf{d}t^2 + \frac{4m}{r}\,\mathbf{d}t\,\mathbf{d}r + \left(1 + \frac{2m}{r}\right)\mathbf{d}r^2 + r^2\mathbf{d}\theta^2 + r^2\sin^2\theta\,\mathbf{d}\varphi^2$$



$\mathscr{H}$: $r = 2m$; coord: (t, θ, φ)

$\mathscr{S}$: $r = 2m$ and $t = t_0$; coord: (θ, φ)

$\Longrightarrow$ induced metric on $\mathscr{S}$:

$$\boldsymbol{q} = (2m)^2\left(\mathbf{d}\theta^2 + \sin^2\theta\,\mathbf{d}\varphi^2\right)$$

$$\Longrightarrow q := \det(q_{ab}) = (2m)^4\sin^2\theta$$

$$\Longrightarrow A = \int_{\mathscr{S}} (2m)^2 \sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi$$

$$\Longrightarrow \boxed{A = 16\pi m^2}$$