Geometry of Killing horizons and applications to black hole physics

2. Killing horizons

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https://relativite.obspm.fr/blackholes/ihp24/

Quantum and classical fields interacting with geometry
Institut Henri Poincaré, Paris, France
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1. Null hypersurfaces and non-expanding horizons *(today)*
2. Killing horizons *(today)*
3. Stationary black holes *(tomorrow)*
4. Degenerate Killing horizons and their near-horizon geometry *(tomorrow)*
5. Exploring the extremal Kerr near-horizon geometry with SageMath *(on Thursday)*

Prerequisite

An introductory course on general relativity

Home page for the lectures

https://relativite.obspm.fr/blackholes/ihp24/

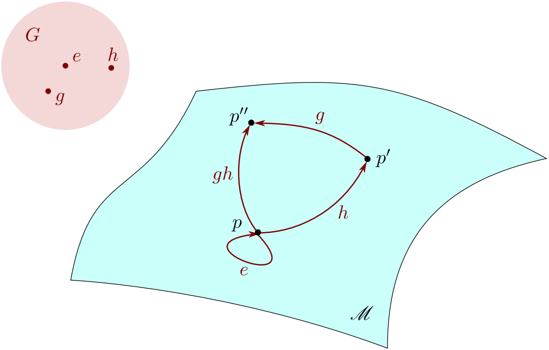
includes

- these slides
- the lecture notes (draft)
- some SageMath notebooks

Lecture 2: Killing horizons

Outline

Group action on spacetime



Given a group G, a **group action of G on $\mathscr{M}$** is a map

$$\begin{array}{rccc} \Phi: & G \times \mathscr{M} & \longrightarrow & \mathscr{M} \\ & (g, p) & \longmapsto & \Phi_g(p) \end{array}$$

such that[a]

- $\forall p \in \mathscr{M}, \ \Phi_e(p) = p$
- $\forall (g,h) \in G^2, \forall p \in \mathscr{M},$ $\Phi_g(\Phi_h(p)) = \Phi_{gh}(p)$

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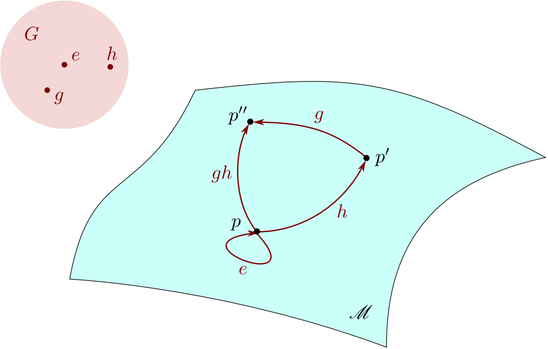
[a] $e :=$ identity element of G

Group action on spacetime



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[a] $e :=$ identity element of G

Orbit of a point $p\in\mathscr{M}$: $\mathrm{orb}_G\, p := \{\Phi_g(p),\ g\in G\}\subset\mathscr{M}$

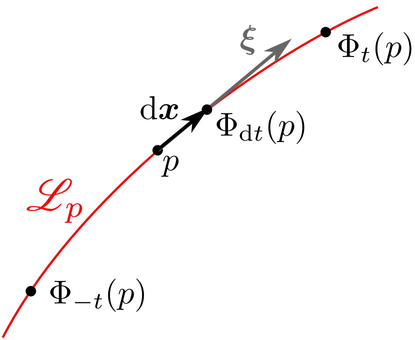
$p =$ **fixed point** of the group action $\iff$ $\mathrm{orb}_G\, p = \{p\}$

Action of a 1-dimensional Lie group

G = 1-dimensional Lie group, parametrized by $t \in \mathbb{R}$ with $g_{t=0} = e$

Notation: $\Phi_t(p) := \Phi_{g_t}(p)$

Either $\mathrm{orb}_G\, p = \{p\}$ (p fixed point) or $\mathrm{orb}_G\, p =: \mathscr{L}_p$ curve in $\mathscr{M}$.



Generator of the action of G on $\mathscr{M}$: vector field $\boldsymbol{\xi}$ tangent to $\mathscr{L}_p$ parametrized by t:

$$\boldsymbol{\xi} := \left.\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right|_{\mathscr{L}_p}$$

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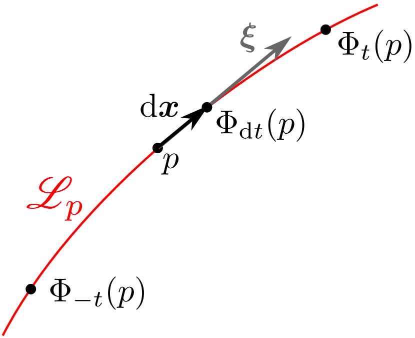

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Infinitesimal displacement under the group action:

$$\overrightarrow{p\,\Phi_{\mathrm{d}t}(p)} = \mathrm{d}t\,\boldsymbol{\xi}$$

Isometries and Killing vectors

A 1-dimensional Lie group G is an **isometry group** of $(\mathscr{M}, \boldsymbol{g})$ iff there is an action Φ of G on $\mathscr{M}$ such that for any value of G's parameter t, Φ_t is an **isometry** of $(\mathscr{M}, \boldsymbol{g})$:

$$\forall p \in \mathscr{M},\ \forall(\boldsymbol{u}, \boldsymbol{v}) \in (T_p\mathscr{M})^2, \quad \left.\boldsymbol{g}\right|_{\Phi_t(p)}(\Phi_{t*}\boldsymbol{u}, \Phi_{t*}\boldsymbol{v}) = \left.\boldsymbol{g}\right|_p(\boldsymbol{u}, \boldsymbol{v}),$$

where $\Phi_{t*}\boldsymbol{u}$ stands for the pushforward of $\boldsymbol{u}$ by Φ_t.

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where $\Phi_{t*}\boldsymbol{u}$ stands for the pushforward of $\boldsymbol{u}$ by Φ_t.

By def. of the pullback $\Phi_t^*\boldsymbol{g}$ of $\boldsymbol{g}$ by Φ_t, $\boldsymbol{g}|_{\Phi_t(p)} (\Phi_{t*}\boldsymbol{u}, \Phi_{t*}\boldsymbol{v}) =: \Phi_t^*\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v})$.

Hence Φ_t isometry $\iff \Phi_t^*\boldsymbol{g} = \boldsymbol{g}$. In view of the definition of the Lie derivative $\mathcal{L}_{\boldsymbol{\xi}}\, \boldsymbol{g} := \lim_{t\to 0} \frac{1}{t} \left(\Phi_t^*\boldsymbol{g} - \boldsymbol{g}\right)$, we conclude

Characterization of continuous spacetime isometries

A 1-dimensional Lie group G, of generator $\boldsymbol{\xi}$, is an isometry group of $(\mathscr{M}, \boldsymbol{g})$ iff

$$\boxed{\mathcal{L}_{\boldsymbol{\xi}}\, \boldsymbol{g} = 0}$$

The vector field $\boldsymbol{\xi}$ is then called a **Killing vector** of $(\mathscr{M}, \boldsymbol{g})$, the above equation being equivalent to the **Killing equation**:

$$\boxed{\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0}$$

Outline

Killing horizons

Definition

A **Killing horizon** is a connected null hypersurface $\mathscr{H}$ in a spacetime $(\mathscr{M}, \boldsymbol{g})$ endowed with a Killing vector $\boldsymbol{\xi}$ such that, on $\mathscr{H}$, $\boldsymbol{\xi}$ is normal to $\mathscr{H}$.

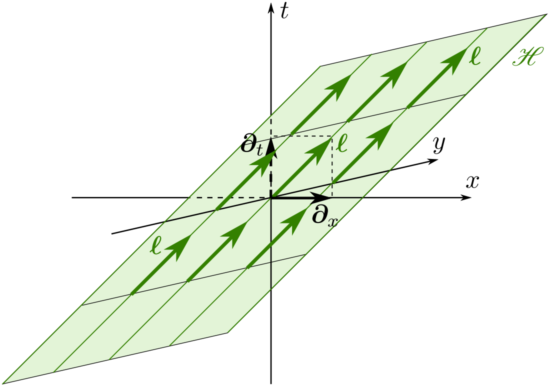
$\Longrightarrow$ $\boldsymbol{\xi}|_{\mathscr{H}} \neq 0$ and $\boldsymbol{\xi}$ is null on $\mathscr{H}$

Equivalent definition

A **Killing horizon** is a connected null hypersurface $\mathscr{H}$ whose null geodesic generators are orbits of a 1-parameter group of isometries of $(\mathscr{M}, \boldsymbol{g})$.

$\Longrightarrow$ $\mathscr{H}$ stable (globally invariant) by the group action

Example 1: null hyperplane in Minkowski spacetime as a translation Killing horizon



$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$

$\mathscr{H}: \ u := t - x = 0$

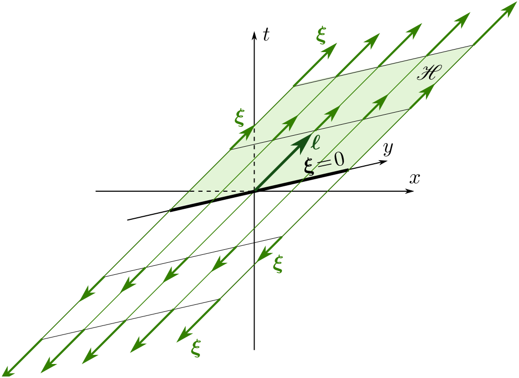
$\boldsymbol{\ell} = \boldsymbol{\partial}_t + \boldsymbol{\partial}_x$

$G = (\mathbb{R}, +)$ acting by translations in the direction $\boldsymbol{\partial}_t + \boldsymbol{\partial}_x$

Killing vector: $\boldsymbol{\xi} = \boldsymbol{\partial}_t + \boldsymbol{\partial}_x$

$\boldsymbol{\xi} \stackrel{\mathscr{H}}{=} \boldsymbol{\ell}$

Example 2: null half-hyperplane in Minkowski spacetime as a boost Killing horizon



$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$

$\mathscr{H}$: $u := t - x = 0$ and $t > 0$

null normal: $\boldsymbol{\ell} = \boldsymbol{\partial}_t + \boldsymbol{\partial}_x$

$G = (\mathbb{R}, +)$ acting by Lorentz boosts[a] in the (t, x) plane

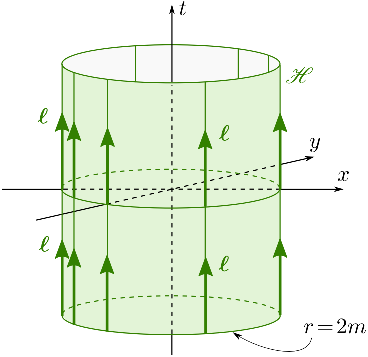
Killing vector: $\boldsymbol{\xi} = x\boldsymbol{\partial}_t + t\boldsymbol{\partial}_x$

$\boldsymbol{\xi} \overset{\mathscr{H}}{=} t(\boldsymbol{\partial}_t + \boldsymbol{\partial}_x) \overset{\mathscr{H}}{=} t\,\boldsymbol{\ell}$

[a] Parameter of G: boost rapidity

Example 3: the Schwarzschild horizon

$$\boldsymbol{g} = -\left(1 - \frac{2m}{r}\right)\mathbf{d}t^2 + \frac{4m}{r}\,\mathbf{d}t\,\mathbf{d}r + \left(1 + \frac{2m}{r}\right)\mathbf{d}r^2 + r^2\mathbf{d}\theta^2 + r^2\sin^2\theta\,\mathbf{d}\varphi^2$$



$\mathscr{H}$: $r = 2m$

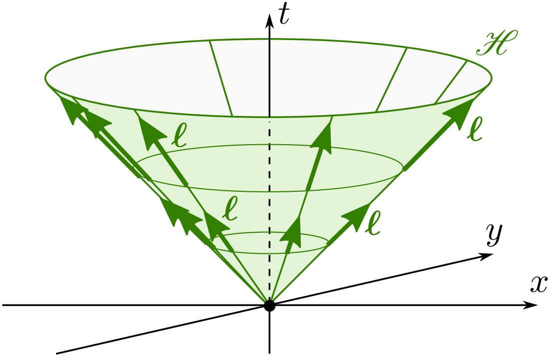
null normal: $\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{r-2m}{r+2m}\,\boldsymbol{\partial}_r$

$G = (\mathbb{R}, +)$ acting by (time) translation (stationarity)

Killing vector: $\boldsymbol{\xi} = \boldsymbol{\partial}_t$

$\boldsymbol{\xi} \stackrel{\mathscr{H}}{=} \boldsymbol{\ell}$

Counter-example 1: future null cone in Minkowski spacetime



$$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$$

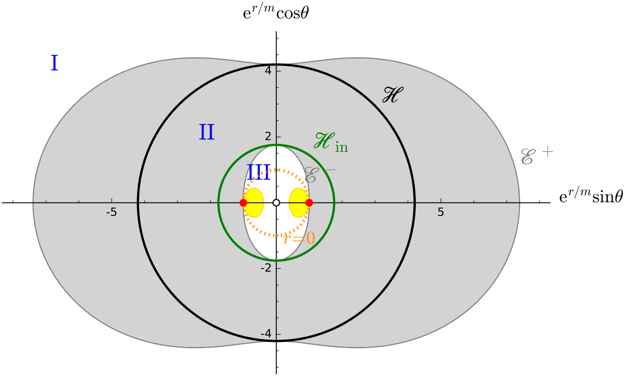
$$\mathscr{H}:\ t - \sqrt{x^2 + y^2 + z^2} = 0$$

null normal:

$$\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{x}{r}\boldsymbol{\partial}_x + \frac{y}{r}\boldsymbol{\partial}_y + \frac{z}{r}\boldsymbol{\partial}_z$$

$\mathscr{H}$ is globally invariant under the action of the Lorentz group $\mathrm{O}(3,1)$, but its null generators are not invariant under the action of a single 1-dimensional subgroup of $\mathrm{O}(3,1)$.

Counter-example 2: Kerr ergosphere



Kerr spacetime $(\mathscr{M}, \boldsymbol{g})$

- mass m
- specific angular momentum $a > 0$

Boyer-Lindquist coord. (t, r, θ, φ)

- outer ergosphere: $\mathscr{E}^+$: $r = m + \sqrt{m^2 - a^2 \cos^2\theta}$
- stationary Killing vector: $\boldsymbol{\xi} = \boldsymbol{\partial}_t$

On $\mathscr{E}^+$, $\boldsymbol{\xi}$ is null and tangent to $\mathscr{E}^+$. However, $\mathscr{E}^+$ is *not* a Killing horizon for $\boldsymbol{\xi}$ is not normal to $\mathscr{E}^+$. Actually $\mathscr{E}^+$ is a timelike hypersurface for $a \neq 0$.

Outline

Vanishing of the deformation rate tensor

Vanishing of the deformation rate tensor

On a Killing horizon $\mathscr{H}$, the deformation rate $\boldsymbol{\Theta}$ of any cross-section $\mathscr{S}$ along the null normal $\boldsymbol{\ell} \overset{\mathscr{H}}{=} \boldsymbol{\xi}$ vanishes:

$$\boldsymbol{\Theta} = 0$$

Proof: by definition (cf. Lecture 1), $\boldsymbol{\Theta} := \frac{1}{2}\overrightarrow{\boldsymbol{q}}^{*}\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\,\boldsymbol{q}$, where $\boldsymbol{q}$ is the metric induced by $\boldsymbol{g}$ on $\mathscr{S}$ and $\overrightarrow{\boldsymbol{q}}^{*}$ is the operator making $\boldsymbol{\Theta}$ a spacetime tensor by orthogonal projection onto $\mathscr{S}$: $\Theta_{\alpha\beta} = \frac{1}{2} q^{\mu}{}_{\alpha} q^{\nu}{}_{\beta} \mathcal{L}_{\boldsymbol{\ell}}\, q_{\mu\nu}$. Since $\boldsymbol{\ell} \overset{\mathscr{H}}{=} \boldsymbol{\xi}$ we get $\boldsymbol{\Theta} = \frac{1}{2}\overrightarrow{\boldsymbol{q}}^{*}\boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}}\,\boldsymbol{q}$ with $\boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}}\,\boldsymbol{q} = 0$ since $\boldsymbol{\xi}$ is an isometry generator. □

Corollary

On a Killing horizon $\mathscr{H}$, the expansion along any null normal $\boldsymbol{\ell}$ vanishes:

$$\theta_{(\boldsymbol{\ell})} = 0$$

Proof: $\theta_{(\boldsymbol{\ell})} = q^{ab}\Theta_{ab} = 0$. □

Killing horizons as non-expanding horizons

Recall the definition (cf. Lecture 1):

A **non-expanding horizon** is a null hypersurface with compact complete cross-sections and vanishing expansion: $\theta_{(\boldsymbol{\ell})} = 0$

Hence:

A Killing horizon with compact complete cross-sections is a non-expanding horizon.

Example: the Schwarzschild horizon

Outline

Surface gravity of a Killing horizon

Definition

Let $\mathscr{H}$ be a Killing horizon w.r.t. a Killing vector $\boldsymbol{\xi}$. The non-affinity coefficient κ of $\boldsymbol{\xi}$ considered as a null normal to $\mathscr{H}$, i.e. the coefficient κ such that (cf. Lecture 1)

$$\boldsymbol{\nabla}_{\boldsymbol{\xi}}\boldsymbol{\xi} \stackrel{\mathscr{H}}{=} \kappa\,\boldsymbol{\xi},$$

is called the **surface gravity of $\mathscr{H}$**.

Surface gravity of a Killing horizon

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is called the **surface gravity of $\mathscr{H}$**.

Thanks to the Killing equation, we have

$$\kappa\xi_\alpha \stackrel{\mathscr{H}}{=} \xi^\mu\nabla_\mu\xi_\alpha = -\xi^\mu\nabla_\alpha\xi_\mu = -\frac{1}{2}\nabla_\alpha(\xi_\mu\xi^\mu)$$

Hence:

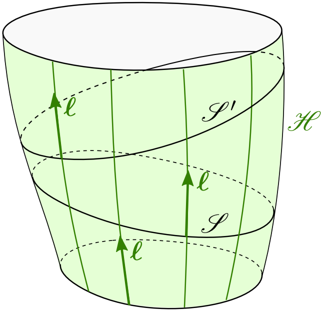
$$\mathbf{d}(\boldsymbol{\xi}\cdot\boldsymbol{\xi}) \stackrel{\mathscr{H}}{=} -2\kappa\,\underline{\boldsymbol{\xi}}$$

Explicit expression of the surface gravity

$$\kappa^2 \stackrel{\mathscr{H}}{=} -\frac{1}{2}\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu$$

Proof: Since $\boldsymbol{\xi}$ is hypersurface-orthogonal on $\mathscr{H}$, the Frobenius theorem implies that there exists a scalar field a on $\mathscr{H}$ such that $\mathbf{d}\underline{\boldsymbol{\xi}} \stackrel{\mathscr{H}}{=} \boldsymbol{a} \wedge \underline{\boldsymbol{\xi}}$, i.e. $\nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha \stackrel{\mathscr{H}}{=} a_\alpha\, \xi_\beta - a_\beta\, \xi_\alpha$. Thanks to the Killing equation, this yields $2\nabla_\alpha \xi_\beta \stackrel{\mathscr{H}}{=} a_\alpha\, \xi_\beta - a_\beta\, \xi_\alpha$. Contracting with ξ^α and using $\xi^\mu \xi_\mu \stackrel{\mathscr{H}}{=} 0$ and $\xi^\mu \nabla_\mu \xi_\beta = \kappa \xi_\beta$, we get $a_\mu \xi^\mu \stackrel{\mathscr{H}}{=} 2\kappa$. Then expanding $4\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu \stackrel{\mathscr{H}}{=} (a_\mu \xi_\nu - a_\nu \xi_\mu)(a^\mu \xi^\nu - a^\nu \xi^\mu)$ yields the result. □

Variation of κ over $\mathscr{H}$ (1/3)



It is obvious that κ must be constant along any null geodesic generator $\mathscr{L}$ of the Killing horizon $\mathscr{H}$: $\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \kappa = \boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}}\, \kappa = 0$ since ξ is a spacetime symmetry generator and κ is defined solely from $\boldsymbol{\xi}$. But, a priori, κ could vary from one generator of $\mathscr{H}$ to the other. To conclude about this last point, it suffices to study the variation of κ along a cross-section $\mathscr{S}$ of $\mathscr{H}$.

As for the Raychaudhuri equation (cf. Lecture 1), start from the contracted Ricci equation for the null normal $\boldsymbol{\ell} \stackrel{\mathscr{H}}{=} \boldsymbol{\xi}$:

$$\nabla_\mu \nabla_\alpha \ell^\mu - \nabla_\alpha \nabla_\mu \ell^\mu = R_{\mu\alpha} \ell^\mu$$

Contract it with a generic tangent vector to $\mathscr{S}$, $\boldsymbol{v}$ say. Using $\nabla_\alpha \ell^\mu = \Theta_\alpha{}^\mu + \omega_\alpha \ell^\mu - \ell_\alpha k^\nu \nabla_\nu \ell^\mu$ with $\Theta_\alpha{}^\mu = 0$ and $\nabla_\mu \ell^\mu = \theta_{(\boldsymbol{\ell})} + \kappa = \kappa$, one gets (cf. Sec. 3.3.5 of the lecture notes for details)

$$\langle \boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \boldsymbol{\omega}, \boldsymbol{v} \rangle - \boldsymbol{\nabla}_{\boldsymbol{v}}\, \kappa = \boldsymbol{R}(\boldsymbol{\ell}, \boldsymbol{v})$$

Variation of κ over $\mathscr{H}$ (2/3)

Since both $\boldsymbol{v}$ and $\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \boldsymbol{v}$ are tangent to $\mathscr{H}$, we can write $\langle \boldsymbol{\omega}, \boldsymbol{v}\rangle = \langle {}^{\mathscr{H}}\boldsymbol{\omega}, \boldsymbol{v}\rangle$ and $\langle \boldsymbol{\omega}, \boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \boldsymbol{v}\rangle = \langle {}^{\mathscr{H}}\boldsymbol{\omega}, \boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \boldsymbol{v}\rangle$, where ${}^{\mathscr{H}}\boldsymbol{\omega}$ is the connection 1-form introduced on non-expanding horizons with $\boldsymbol{\Theta} = 0$ in Lecture 1. Then, by means of the Leibnitz rule,

$$\langle \boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \boldsymbol{\omega}, \boldsymbol{v}\rangle = \boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \underbrace{\langle \boldsymbol{\omega}, \boldsymbol{v}\rangle}_{\langle {}^{\mathscr{H}}\boldsymbol{\omega}, \boldsymbol{v}\rangle} - \underbrace{\langle \boldsymbol{\omega}, \boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \boldsymbol{v}\rangle}_{\langle {}^{\mathscr{H}}\boldsymbol{\omega}, \boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \boldsymbol{v}\rangle} = \langle \boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, {}^{\mathscr{H}}\boldsymbol{\omega}, \boldsymbol{v}\rangle$$

Now, since ${}^{\mathscr{H}}\boldsymbol{\omega}$ is a geometry quantity intrinsic to $\mathscr{H}$, one has $\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, {}^{\mathscr{H}}\boldsymbol{\omega} = \boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}}\, {}^{\mathscr{H}}\boldsymbol{\omega} = 0$. Hence

$$\boldsymbol{\nabla}_{\boldsymbol{v}}\, \kappa = -\boldsymbol{R}(\boldsymbol{\ell}, \boldsymbol{v})$$

To go further, we shall set some condition on the Ricci tensor...

Null dominance condition

Null dominance condition

$\boldsymbol{G} := \boldsymbol{R} - (R/2)\boldsymbol{g}$ being the Einstein tensor of $\boldsymbol{g}$, there exists a scalar field f such that for any future-directed null vector $\boldsymbol{\ell}$, the vector

$$\boldsymbol{W} := -\overrightarrow{\boldsymbol{G}}(\boldsymbol{\ell}) - f\boldsymbol{\ell} \iff W^\alpha := -G^\alpha_{\ \ \mu}\ell^\mu - f\ell^\alpha$$

is zero or future-directed (null or timelike).

Remark: The null dominance condition implies the null convergence condition $\boldsymbol{R}(\boldsymbol{\ell},\boldsymbol{\ell}) \geq 0$ (cf. Lecture 1). Indeed, since $\boldsymbol{g}(\boldsymbol{\ell},\boldsymbol{\ell}) = 0$, we have $\boldsymbol{R}(\boldsymbol{\ell},\boldsymbol{\ell}) = \boldsymbol{G}(\boldsymbol{\ell},\boldsymbol{\ell}) + f\boldsymbol{g}(\boldsymbol{\ell},\boldsymbol{\ell}) = -\boldsymbol{W}\cdot\boldsymbol{\ell} \geq 0$ for both $\boldsymbol{W}$ and $\boldsymbol{\ell}$ are future-directed.

Null dominant energy condition

If general relativity is assumed, the null dominance condition is implied with $f = \Lambda$ (cosmological constant) by the following energy condition:

Null dominant energy condition

$\boldsymbol{T}$ being the energy-momentum tensor and $\boldsymbol{\ell}$ being any future-directed null vector, the vector

$$\boldsymbol{W} := -\vec{\boldsymbol{T}}(\boldsymbol{\ell}) \iff W^\alpha = -T^\alpha_{\ \ \mu}\ell^\mu$$

is zero or future-directed (null or timelike).

By continuity, the null dominant energy condition is implied by the

Dominant energy condition

$\boldsymbol{T}$ being the energy-momentum tensor and $\boldsymbol{u}$ being any future-directed timelike vector, the vector

$$\boldsymbol{W} := -\vec{\boldsymbol{T}}(\boldsymbol{u}) \iff W^\alpha := -T^\alpha_{\ \ \mu}u^\mu$$

is zero or future-directed (null or timelike).

Physically: the energy flux is causal.

Variation of κ over $\mathscr{H}$ (3/3)

Let us assume the null dominance condition and come back to the result

$$\nabla_{\boldsymbol{v}}\, \kappa = -\boldsymbol{R}(\boldsymbol{\ell}, \boldsymbol{v})$$

For $\boldsymbol{W} := -\vec{\boldsymbol{G}}(\boldsymbol{\ell}) - f\boldsymbol{\ell}$, we have

$\boldsymbol{W} \cdot \boldsymbol{v} = -\boldsymbol{R}(\boldsymbol{\ell}, \boldsymbol{v}) + \left(\frac{R}{2} - f\right) \underbrace{\boldsymbol{\ell} \cdot \boldsymbol{v}}_{0} = -\boldsymbol{R}(\boldsymbol{\ell}, \boldsymbol{v})$. Hence

$$\nabla_{\boldsymbol{v}}\, \kappa = \boldsymbol{W} \cdot \boldsymbol{v}$$

Now, $\boldsymbol{\ell} \cdot \boldsymbol{W} = \boldsymbol{R}(\boldsymbol{\ell}, \boldsymbol{\ell}) \overset{\mathscr{H}}{=} 0$ by the null convergence condition (implied by the null dominance) + the null Raychaudhuri equation (cf. Lecture 1). It follows that $\boldsymbol{W}$ is tangent to $\mathscr{H}$. Since $\mathscr{H}$ is null, $\boldsymbol{W}$ must be either zero, spacelike or null and collinear to $\boldsymbol{\ell}$. By the null dominance condition, $\boldsymbol{W}$ cannot be spacelike. We have thus $\boldsymbol{W} = \alpha \boldsymbol{\ell}$ with $\alpha \geq 0$. It follows immediately that $\boldsymbol{W} \cdot \boldsymbol{v} = \alpha \boldsymbol{\ell} \cdot \boldsymbol{v} = 0$, so that we are left with

$$\nabla_{\boldsymbol{v}}\, \kappa = 0$$

We conclude that κ is constant over $\mathscr{S}$ and thus over $\mathscr{H}$.

Zeroth law of black hole dynamics

We have thus obtained the

Zeroth law of black hole dynamics (Hawking 1973, Carter 1973)

If the *null dominance condition* is fulfilled on a Killing horizon $\mathscr{H}$ — which is guaranteed in general relativity if the *null dominant energy condition* holds —, then the surface gravity κ is uniform over $\mathscr{H}$:

$$\boxed{\kappa = \mathrm{const.}}$$

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$\Longrightarrow$ Analogy with the Zeroth law of thermodynamics: the temperature T of a body in equilibrium is uniform over the body

Hawking's rigidity theorem (cf. Lecture 3): in (electro)vacuum general relativity, the event horizon of a black hole in equilibrium (stationary spacetime) is a Killing horizon

Hawking radiation: $T = \dfrac{\kappa}{2\pi}$

Examples

- Null hyperplane in Minkowski spacetime as a translation Killing horizon (Example 1 above): $\kappa = 0$
- Null half-hyperplane in Minkowski spacetime as a boost Killing horizon (Example 2 above): $\kappa = 1$
- Schwarzschild horizon (Example 3 above): $\kappa = \dfrac{1}{4m}$
- Kerr horizon: $\kappa = \dfrac{\sqrt{m^2 - a^2}}{2m(m + \sqrt{m^2 - a^2})}$

In all the above examples, the null dominance condition is trivially fulfilled since $\boldsymbol{G} = 0$.

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Counter-example: The surface gravity of rotating stationary black holes in the *cubic Galileon* scalar-tensor theory of gravity is not constant [Grandclément, CQG **41**, 025012 (2024)]. This evades the Zeroth law because the null dominance condition is not satisfied by these solutions.

Classification of Killing horizons

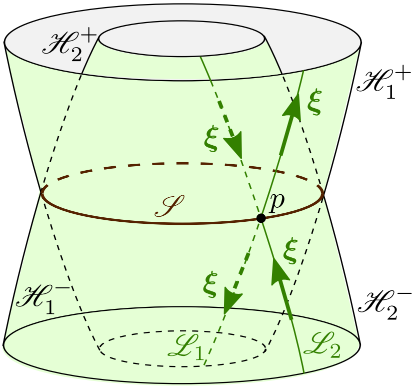
Since κ is constant (assuming the null dominance condition), Killing horizons can be classified in two types:

- **degenerate Killing horizon**: $\kappa = 0$;
  the Killing vector $\boldsymbol{\xi}$ is a geodesic vector on $\mathscr{H}$
- **non-degenerate Killing horizon**: $\kappa \neq 0$;
  the Killing vector $\boldsymbol{\xi}$ is only a pregeodesic vector on $\mathscr{H}$

Outline

Bifurcate Killing horizons

$(\mathscr{M}, \boldsymbol{g}) = n$-dimensional spacetime endowed with a Killing vector field $\boldsymbol{\xi}$



A **bifurcate Killing horizon** is the union

$$\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2,$$

where

- $\mathscr{H}_1$ and $\mathscr{H}_2$ are two null hypersurfaces;
- $\mathscr{S} := \mathscr{H}_1 \cap \mathscr{H}_2$ is a spacelike $(n-2)$-surface;
- each of the sets $\mathscr{H}_1 \setminus \mathscr{S}$ and $\mathscr{H}_2 \setminus \mathscr{S}$ has two connected components, which are Killing horizons w.r.t. $\boldsymbol{\xi}$.

The $(n-2)$-dimensional submanifold $\mathscr{S}$ is called the **bifurcation surface** of $\mathscr{H}$.

A first property of bifurcate Killing horizons

Vanishing of the Killing vector at the bifurcation surface

The Killing vector field vanishes at the bifurcation surface of a bifurcate Killing horizon:

$$\boldsymbol{\xi}|_{\mathscr{S}} = 0$$

A first property of bifurcate Killing horizons

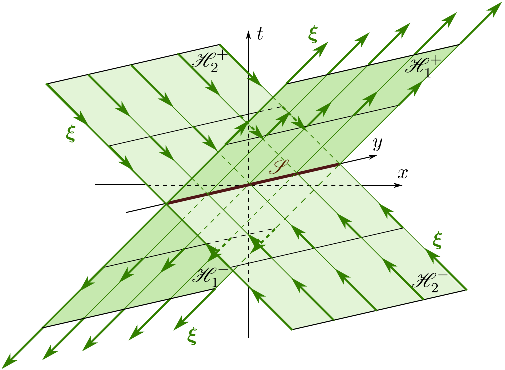
Vanishing of the Killing vector at the bifurcation surface

The Killing vector field vanishes at the bifurcation surface of a bifurcate Killing horizon:

$$\boldsymbol{\xi}|_{\mathscr{S}} = 0$$

Proof: Let $p \in \mathscr{S}$ and let us assume that $\boldsymbol{\xi}|_p \neq 0$. Let $\mathscr{L}_1$ (resp. $\mathscr{L}_2$) be the null geodesic generator of $\mathscr{H}_1$ (resp. $\mathscr{H}_2$) that intersects $\mathscr{S}$ at p. By definition of a Killing horizon, $\boldsymbol{\xi}$ is tangent to $\mathscr{L}_1 \cap \mathscr{H}_1^+$ and to $\mathscr{L}_1 \cap \mathscr{H}_1^-$, i.e. to $\mathscr{L}_1 \setminus \{p\}$. If $\boldsymbol{\xi}|_p \neq 0$, then by continuity, $\boldsymbol{\xi}$ is a (non-vanishing) tangent vector field all along $\mathscr{L}_1$. Similarly, $\boldsymbol{\xi}$ is tangent to all $\mathscr{L}_2$. At their intersection point p, the geodesics $\mathscr{L}_1$ and $\mathscr{L}_2$ have thus a common tangent vector, namely $\boldsymbol{\xi}|_p$. The geodesic uniqueness theorem then implies $\mathscr{L}_1 = \mathscr{L}_2$, so that $\mathscr{L}_1 \subset \mathscr{H}_1 \cap \mathscr{H}_2 = \mathscr{S}$. But since $\mathscr{S}$ is spacelike and $\mathscr{L}_1$ is null, we reach a contradiction. Hence we must have $\boldsymbol{\xi}|_p = 0$. □

Example 1: bifurcate Killing horizon w.r.t. a Lorentz boost generator



$(\mathscr{M}, \boldsymbol{g})$: Minkowski spacetime

$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$

Killing vector: $\boldsymbol{\xi} = x\boldsymbol{\partial}_t + t\boldsymbol{\partial}_x$

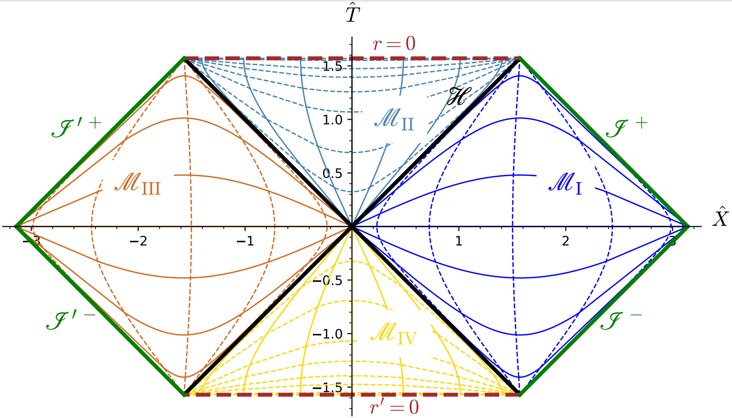
$\Longrightarrow$ generates Lorentz boosts in the plane (t, x)

$\mathscr{H}_1$: $t = x$

$\mathscr{H}_2$: $t = -x$

$\mathscr{S}$: $(t, x) = (0, 0)$

Example 2: bifurcate Killing horizon in Schwarzschild spacetime

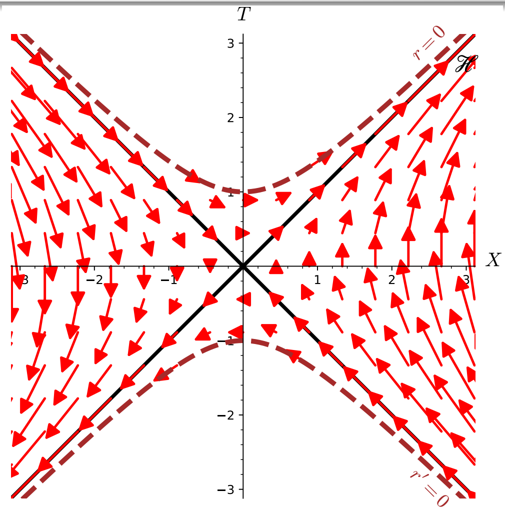


Dashed lines: field lines of the stationary Killing vector ξ

Thick black lines: bifurcate Killing horizon w.r.t. ξ

https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/Schwarz_conformal_std.ipynb

Example 2: bifurcate Killing horizon in Schwarzschild spacetime



Stationary Killing vector $\boldsymbol{\xi}$ in the maximal extension of Schwarzschild spacetime (Kruskal diagram)

https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/Schwarz_Kruskal_Szekeres.ipynb

Affine parametrization of a non-degenerate Killing horizon

Let $\mathscr{H}$ be a Killing horizon w.r.t. a Killing vector $\boldsymbol{\xi}$ of constant surface gravity $\kappa \neq 0$. Let t be the parameter of the null geodesic generators $\mathscr{L}$ of $\mathscr{H}$ associated to $\boldsymbol{\xi}$ ($\boldsymbol{\xi} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ along $\mathscr{L}$). The null vector field $\boldsymbol{\ell}$ defined on $\mathscr{H}$ by

$$\boldsymbol{\ell} = \mathrm{e}^{-\kappa t}\,\boldsymbol{\xi} \quad \Longleftrightarrow \quad \boldsymbol{\xi} = \mathrm{e}^{\kappa t}\,\boldsymbol{\ell}$$

is a geodesic vector field and the affine parameter associated to it is

$$\lambda = \frac{\mathrm{e}^{\kappa t}}{\kappa} + \lambda_0,$$

where λ_0 is constant along a given geodesic generator.

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$$\lambda = \frac{\mathrm{e}^{\kappa t}}{\kappa} + \lambda_0,$$

where λ_0 is constant along a given geodesic generator.

Proof: $$\boldsymbol{\nabla}_{\boldsymbol{\ell}}\,\boldsymbol{\ell} = \boldsymbol{\nabla}_{\mathrm{e}^{-\kappa t}\boldsymbol{\xi}}\left(\mathrm{e}^{-\kappa t}\,\boldsymbol{\xi}\right) = \mathrm{e}^{-\kappa t}\boldsymbol{\nabla}_{\boldsymbol{\xi}}\left(\mathrm{e}^{-\kappa t}\,\boldsymbol{\xi}\right)$$
$$= \mathrm{e}^{-\kappa t}\Big[\big(\underbrace{\boldsymbol{\nabla}_{\boldsymbol{\xi}}\mathrm{e}^{-\kappa t}}_{\mathrm{d}\mathrm{e}^{-\kappa t}/\mathrm{d}t}\big)\boldsymbol{\xi} + \mathrm{e}^{-\kappa t}\underbrace{\boldsymbol{\nabla}_{\boldsymbol{\xi}}\,\boldsymbol{\xi}}_{\kappa\boldsymbol{\xi}}\Big] = 0 \Longrightarrow \boldsymbol{\ell}\ \text{geodesic}$$

Moreover, $\dfrac{\mathrm{d}\lambda}{\mathrm{d}t} = \boldsymbol{\xi}(\lambda) = \mathrm{e}^{\kappa t}\,\underbrace{\boldsymbol{\ell}(\lambda)}_{1} = \mathrm{e}^{\kappa t}$ yields $\lambda = \dfrac{\mathrm{e}^{\kappa t}}{\kappa} + \lambda_0$. □

Incompleteness of the null generators of a non-degenerate Killing horizon

Let us assume $\kappa > 0$ and consider a null geodesic generator $\mathscr{L}$ of $\mathscr{H}$. Parameterize $\mathscr{L}$ by the isometry group parameter t ($\boldsymbol{\xi} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t|_{\mathscr{L}}$).

From $\lambda = \dfrac{\mathrm{e}^{\kappa t}}{\kappa} + \lambda_0$, we get $t \in (-\infty, +\infty) \iff \lambda \in (\lambda_0, +\infty)$. Since λ is an affine parameter of $\mathscr{L}$, this means that $\mathscr{L}$ is an incomplete geodesic. Moreover, one deduces from $\boldsymbol{\xi} = \mathrm{e}^{\kappa t}\,\boldsymbol{\ell}$ that

$$\boldsymbol{\xi} \to 0 \quad \text{when} \quad t \to -\infty \qquad (\kappa > 0)$$

More precisely:

Boyer's theorem (1969)

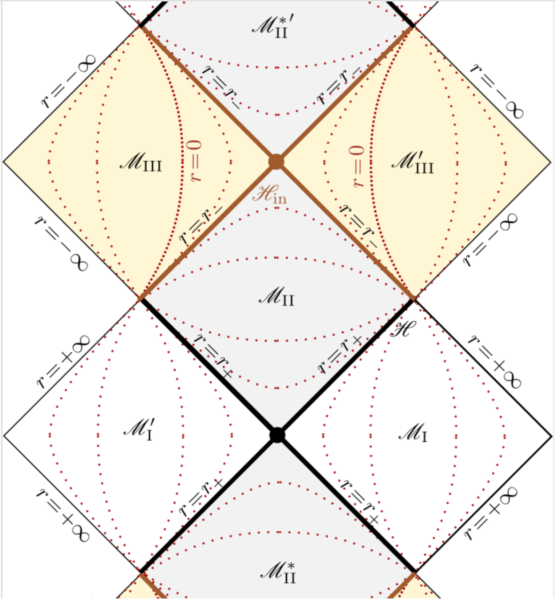
A Killing horizon $\mathscr{H}$ w.r.t a Killing vector $\boldsymbol{\xi}$ is contained in a bifurcate Killing horizon iff $\mathscr{H}$ contains at least one null geodesic orbit of the isometry group that is complete as an orbit (t takes all values in $\mathbb{R}$), but that is incomplete and extendable as a geodesic.

Non-degenerate Killing horizons as part of a bifurcate Killing horizon

One deduces from Boyer's theorem that

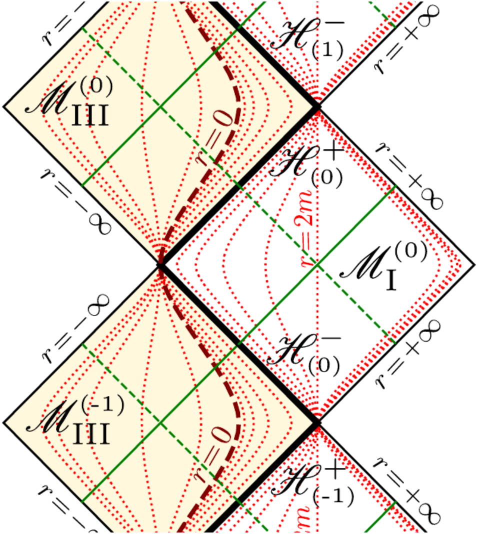
The null geodesic generators of a non-degenerate Killing horizon $\mathscr{H}$ are incomplete; if they can be extended, $\mathscr{H}$ is contained in a bifurcate Killing horizon, the bifurcation surface of which is the past (resp. future) boundary of $\mathscr{H}$ if $\kappa > 0$ (resp. $\kappa < 0$).

Bifurcate Killing horizons in Kerr spacetime with $0 < a < m$



- *black lines:* outer bifurcate Killing horizon $r = r_+ := m + \sqrt{m^2 - a^2}$ Killing vector $\boldsymbol{\xi}_{\rm out} = \boldsymbol{\partial}_t + \Omega_{\rm out}\boldsymbol{\partial}_\varphi$
- *brown lines:* inner bifurcate Killing horizon $r = r_- := m - \sqrt{m^2 - a^2}$ Killing vector $\boldsymbol{\xi}_{\rm in} = \boldsymbol{\partial}_t + \Omega_{\rm in}\boldsymbol{\partial}_\varphi$

No bifurcate Killing horizon for extremal Kerr ($a = m$)

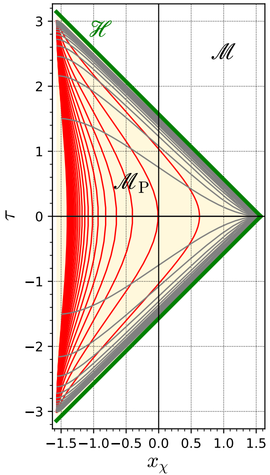


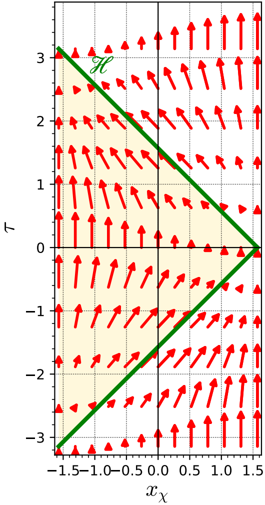
Each thick black line depicts a degenerate Killing horizon ($\kappa = 0$, complete null geodesic generators) at $r = m$.

The intersection of the Killing horizons is a graphical artifact (Carter-Penrose diagram with compactified coordinates): it does not occur in the physical spacetime. Each Killing horizon terminates at an infinite value of the affine parameter of their null geodesic generators. Thus the "intersection" points are actually internal infinities located at $r = m$.

https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/Kerr_extremal_extended.ipynb

Similarity with the Poincaré horizon in AdS_4





Poincaré patch: $\mathscr{M}_{\rm P}$
Poincaré horizon:

$$\mathscr{H} = \partial\mathscr{M}_{\rm P} = \mathscr{H}_+ \cup \mathscr{H}_-$$

$\mathscr{H}_\pm$: degenerate Killing horizon with respect to the Killing vector $\boldsymbol{\xi} = \boldsymbol{\partial}_t$ (in red on the right plot)

$x_\chi = \chi\cos\varphi$,
$\chi \in (0, \pi/2)$, $\varphi \in \{0, \pi\}$

https://nbviewer.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_anti_de_Sitter_Poincare_hor.ipynb

Zeroth law for bifurcate Killing horizons

For a Killing horizon that is part of a bifurcate Killing horizon, one can get the Zeroth law without assuming the null dominance condition:

Zeroth law for bifurcate Killing horizons (Kay & Wald 1991)

The surface gravity is a nonzero constant over any Killing horizon that is part of a bifurcate Killing horizon:

$$\boxed{\kappa = \mathrm{const} \neq 0}$$

Proof: see Sec. 3.4.3 of the lecture notes.

Summary

Main results summarized in an inheritance diagram

Null hypersurface
null geodesic generators
$\nabla_{\boldsymbol{\ell}}\boldsymbol{\ell} = \kappa\boldsymbol{\ell}$

↑

Non-expanding horizon
closed-manifold cross-sections
$\theta_{(\boldsymbol{\ell})} = 0$
area independent of the cross-section
NCC $\Longrightarrow$ $\boldsymbol{\Theta} = 0$
$\Longrightarrow$ induced affine connection

↑

Killing horizon
with closed-manifold cross-sections
$\boldsymbol{\Theta} = 0$
NDC $\Longrightarrow \kappa = \text{const}$ (Zeroth Law 1)
part of BKH $\Longrightarrow \kappa = \text{const} \neq 0$ (Zeroth Law 2)

- ↑ = is a subcase of
- NCC = null convergence condition
- NDC = null dominance condition
- BKH = bifurcate Killing horizon