Basics of black hole physics

1. What is a black hole?

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School on Black Holes and Gravitational Waves
Centre for Strings, Gravitation and Cosmology
Chennai, India
17-22 January 2022

1. What is a black hole? *(today)*
2. Schwarzschild black hole *(tomorrow)*
3. Kerr black hole *(tomorrow)*
4. Black hole dynamics *(on Wednesday)*

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Prerequisite

An introductory course on general relativity

Home page for the lectures

https://luth.obspm.fr/~luthier/gourgoulhon/bh16/chennai/

includes

- these slides
- the lecture notes (draft)
- some SageMath notebooks

Lecture 1: What is a black hole?

Outline

Framework of the lectures

spacetime $= (\mathscr{M}, \boldsymbol{g})$

- $\mathscr{M}$: 4-dimensional smooth manifold
- $\boldsymbol{g}$: Lorentzian metric on $\mathscr{M}$

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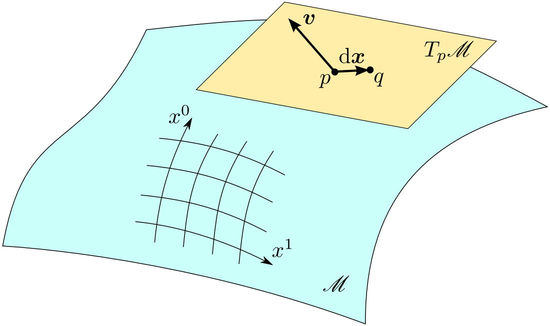

Smooth manifold: topological space $\mathscr{M}$ that *locally* resembles $\mathbb{R}^4$ (but maybe not globally)
$\Longrightarrow$ **coordinate charts**
$\Longrightarrow$ **tangent vectors**

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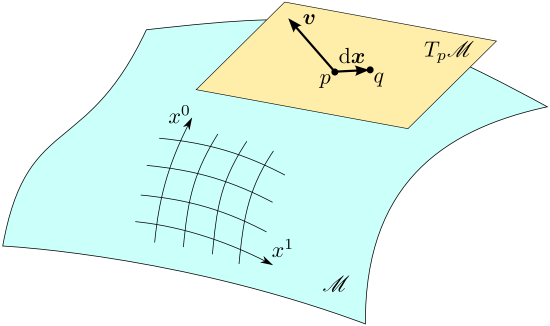

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$\Longrightarrow$ **coordinate charts**
$\Longrightarrow$ **tangent vectors**

Remark: vector connecting two points p and q defined only for p and q infinitely close

Lorentzian metric

Metric tensor $\boldsymbol{g}$: (pseudo) scalar product on $\mathscr{M}$,
i.e. field of nondegenerate symmetric bilinear forms on $\mathscr{M}$:

$$\forall p \in \mathscr{M}, \quad \boldsymbol{g}|_p : \begin{array}{ccc} T_p\mathscr{M} \times T_p\mathscr{M} & \longrightarrow & \mathbb{R} \\ (\boldsymbol{u}, \boldsymbol{v}) & \longmapsto & \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = g_{\mu\nu} u^\mu v^\nu \end{array}$$

of signature $(-, +, +, +)$:

$\exists$ basis $(\boldsymbol{e}_\alpha)_{0\leq\alpha\leq 3}$ of $T_p\mathscr{M}$ such that

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = -u^0 v^0 + u^1 v^1 + u^2 v^2 + u^3 v^3$$

(**Lorentzian** signature)

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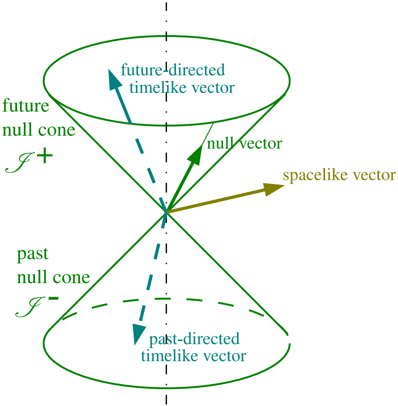
The "line element":

$$\mathrm{d}s^2 := \boldsymbol{g}(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{x}) = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu$$

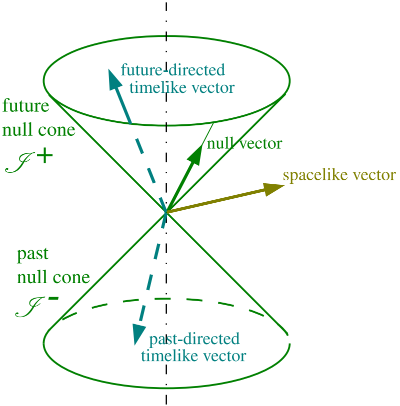
Metric's null cone



Vector $\boldsymbol{v} \in T_p\mathscr{M}$ is

- **spacelike** $\iff$ $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) > 0$
- **null** $\iff$ $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = 0$
- **timelike** $\iff$ $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) < 0$

Metric's null cone



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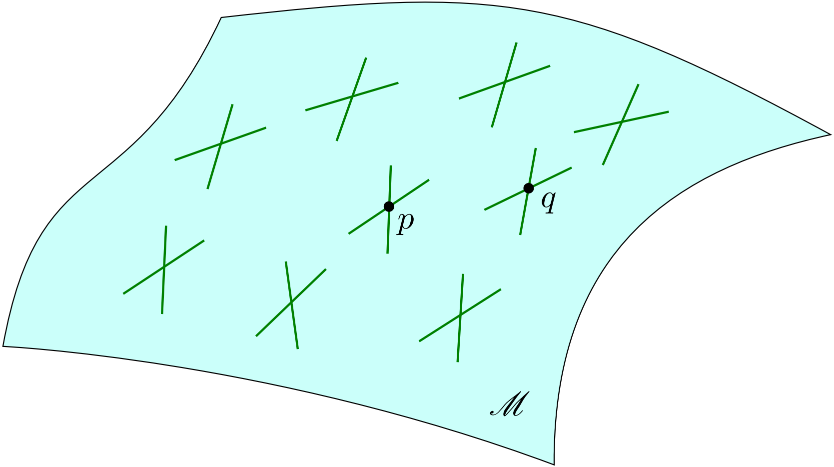
Additional assumption:

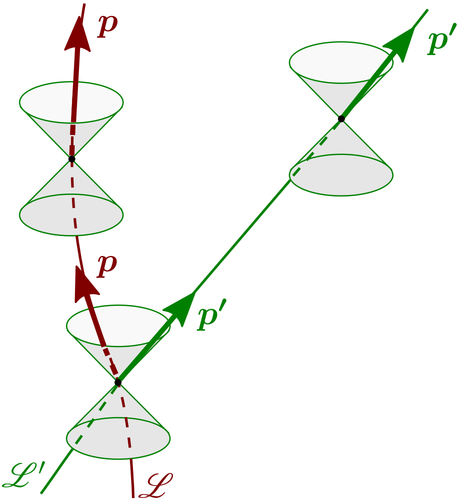
the spacetime $(\mathscr{M}, \boldsymbol{g})$ is **time-oriented**

$\Longrightarrow$ future and past directions

Lorentzian manifold $(\mathscr{M}, \boldsymbol{g})$



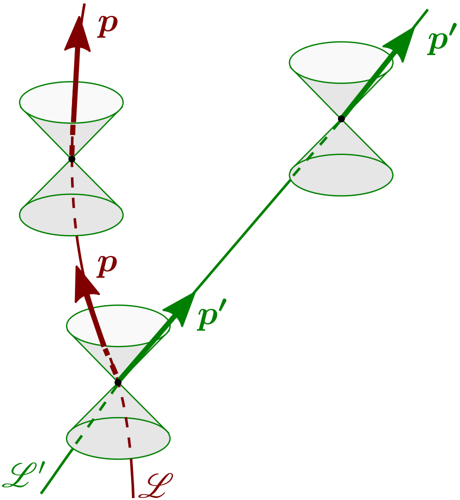
Worldlines



Particle described by its spacetime extent: **worldline** $\mathscr{L}$

massive part. $\iff$ timelike worldline
massless part. $\iff$ null worldline
(tachyon $\iff$ spacelike worldline)

Worldlines



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massless part. $\iff$ null worldline
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Dynamics of a *simple* particle (no spin, no internal structure) entirely described by a future-directed vector field tangent to the worldline: the **4-momentum** $\boldsymbol{p}$

Particle's mass: $m = \sqrt{-\boldsymbol{g}(\boldsymbol{p}, \boldsymbol{p})}$

Einstein's equation

Theory of gravity assumed in these lectures: general relativity

$\Longrightarrow$ the metric tensor $\boldsymbol{g}$ obeys **Einstein's equation:**

$$\boldsymbol{R} - \frac{1}{2} R\,\boldsymbol{g} + \Lambda\,\boldsymbol{g} = 8\pi \boldsymbol{T}$$

where

- $\boldsymbol{R} := \mathrm{Ric}(\boldsymbol{g})$, Ricci tensor: $R_{\alpha\beta} = \mathrm{Riem}(\boldsymbol{g})^{\mu}{}_{\alpha\mu\beta}$
- $R := g^{\mu\nu} R_{\mu\nu}$, Ricci scalar
- Λ cosmological constant
- $\boldsymbol{T}$ energy-momentum tensor of matter/fields

In these lectures: $\Lambda = 0$.

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The definition of a black hole and some of its properties do *not* depend on Einstein's equation.

We shall make clear whether a black hole property relies on Einstein's equation or not.

Outline

What is a black hole?

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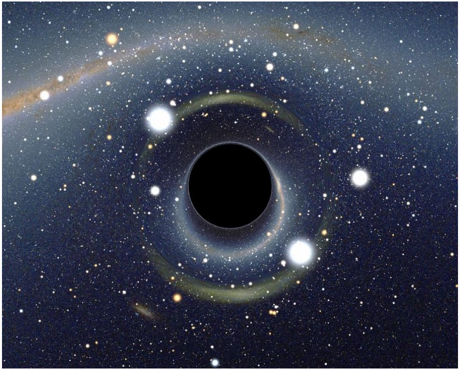
A layperson (loose) definition

A **black hole** is a localized region of spacetime from which neither massive particles nor massless ones (photons) can escape.



[A. Riazuelo, IJMPD **28**, 1950042 (2019)]

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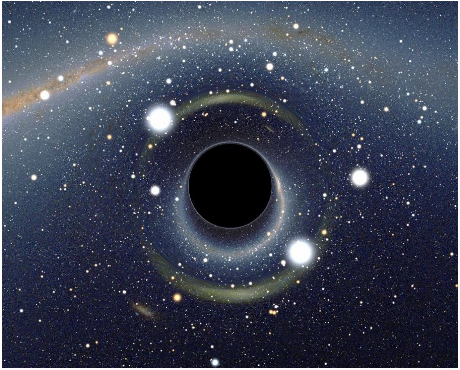

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Two aspects:

- **localization**
- **impassable boundary** (to the exterior)

Impassable boundaries in spacetime

no escape $\Longrightarrow$ black hole region is delimited by an impassable boundary, called the **event horizon**

Boundary in spacetime $\Longrightarrow$ 3-dimensional submanifold, i.e. **hypersurface**

Impassable boundaries in spacetime

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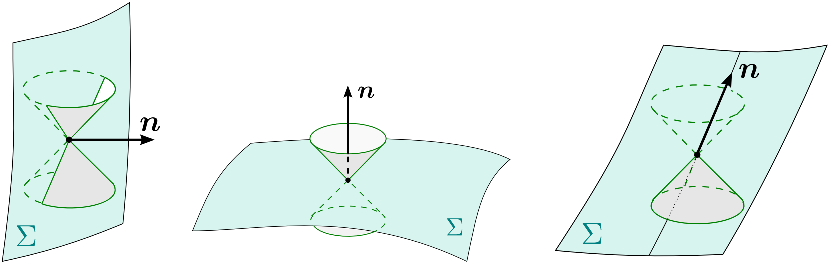
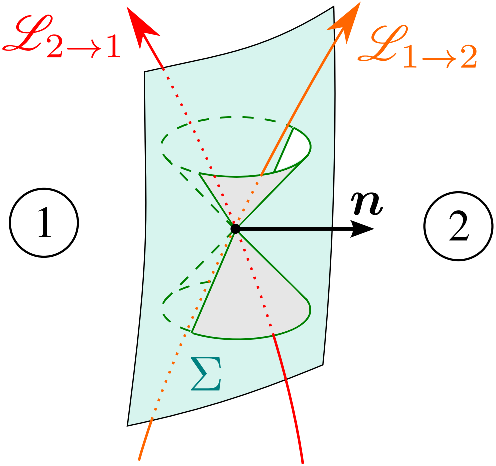
Boundary in spacetime $\Longrightarrow$ 3-dimensional submanifold, i.e. **hypersurface**

Locally, a hypersurface Σ can be of one of 3 types:



Σ timelike	Σ spacelike	Σ null
$\boldsymbol{g}\vert_\Sigma$ Lorentzian	$\boldsymbol{g}\vert_\Sigma$ Riemannian	$\boldsymbol{g}\vert_\Sigma$ degenerate
$\boldsymbol{n}$ spacelike	$\boldsymbol{n}$ timelike	$\boldsymbol{n}$ null (and tangent to Σ)

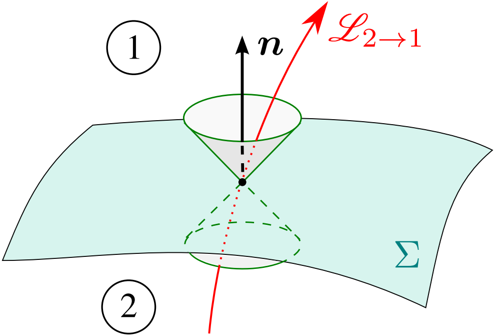
Timelike hypersurface



For worldlines $\mathscr{L}$ directed towards the future:

timelike hypersurface = 2-way membrane
$\Longrightarrow$ not eligible for a black hole boundary

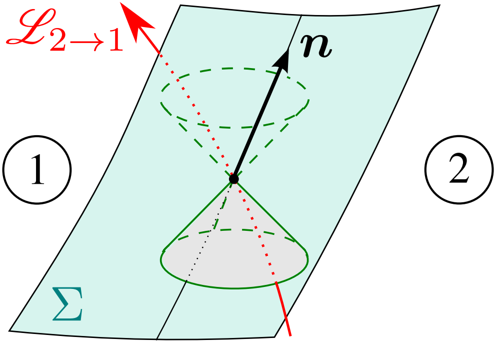
Spacelike hypersurface



For worldlines $\mathscr{L}$ directed towards the future:

spacelike hypersurface = 1-way membrane
$\Longrightarrow$ eligible for a black hole boundary

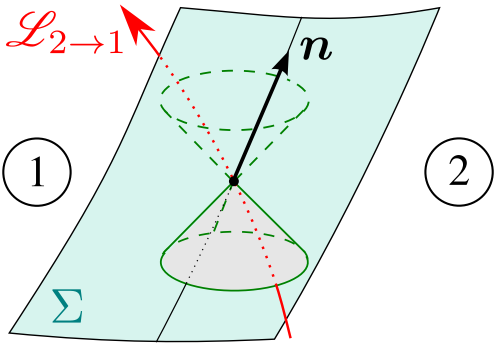
Null hypersurface



For worldlines $\mathscr{L}$ directed towards the future:

null hypersurface = 1-way membrane
$\Longrightarrow$ eligible for a black hole boundary...

Null hypersurface



For worldlines $\mathscr{L}$ directed towards the future:

null hypersurface = 1-way membrane
$\Longrightarrow$ eligible for a black hole boundary...
...and elected! (as a consequence of the formal definition of a black hole, to be given later)

The event horizon of a black hole is a null hypersurface of spacetime.

Outline

Basic geometry of null hypersurfaces

A *generic* hypersurface $\mathscr{H}$ of $(\mathscr{M}, \boldsymbol{g})$ can be (locally) defined as a level set (or "isosurface") of some scalar field $u : \mathscr{M} \to \mathbb{R}$:

$$\mathscr{H} = \{p \in \mathscr{M}, u(p) = 0\}$$

[1] can be turned to $+$ by introducing $u' := -u$

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$$\mathscr{H} = \{p \in \mathscr{M}, u(p) = 0\}$$

Any vector field $\boldsymbol{\ell}$ normal to $\mathscr{H}$ must be collinear to the gradient of u:

$$\boxed{\boldsymbol{\ell} = -\mathrm{e}^{\rho} \vec{\boldsymbol{\nabla}} u}$$

where ρ is some scalar field and the minus sign is chosen for later convenience[1]

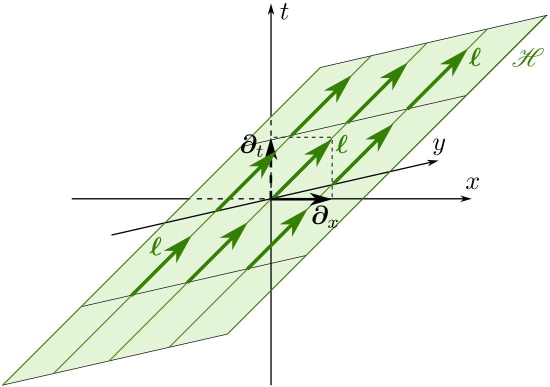
In term of components with respect to a coordinate system (x^α):

$$\ell^\alpha = -\mathrm{e}^{\rho} \nabla^\alpha u = -\mathrm{e}^{\rho} g^{\alpha\mu} \nabla_\mu u = -\mathrm{e}^{\rho} g^{\alpha\mu} \partial_\mu u$$

$\mathscr{H}$ **null** hypersurface $\iff$ $\boldsymbol{g}(\boldsymbol{\ell}, \boldsymbol{\ell}) = 0$ $\iff$ $g^{\mu\nu} \partial_\mu u \, \partial_\nu u = 0$

[1] can be turned to + by introducing $u' := -u$

Example 1: null hyperplane in Minkowski spacetime



$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

$$\boxed{u := t - x = 0}$$

$$\boldsymbol{\nabla} u = \mathbf{d}t - \mathbf{d}x$$

$$\nabla_\alpha u = (1, -1, 0, 0)$$

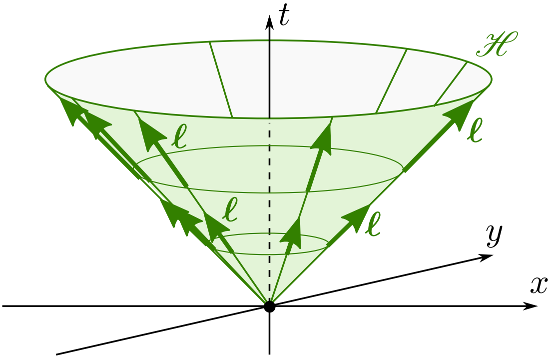
$$\nabla^\alpha u = (-1, -1, 0, 0)$$

Choose $\rho = 0$

$$\Longrightarrow \ell^\alpha = (1, 1, 0, 0)$$

$$\boxed{\boldsymbol{\ell} = \boldsymbol{\partial}_t + \boldsymbol{\partial}_x}$$

Example 2: future null cone in Minkowski spacetime



$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

$$\boxed{u := t - \sqrt{x^2+y^2+z^2} = 0}$$

$$\boldsymbol{\nabla} u = \mathbf{d}t - \frac{x}{r}\mathbf{d}x - \frac{y}{r}\mathbf{d}y - \frac{z}{r}\mathbf{d}z$$

$$r := \sqrt{x^2+y^2+z^2}$$

$$\nabla_\alpha u = \left(1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right)$$

$$\nabla^\alpha u = \left(-1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right)$$

Choose $\rho = 0$

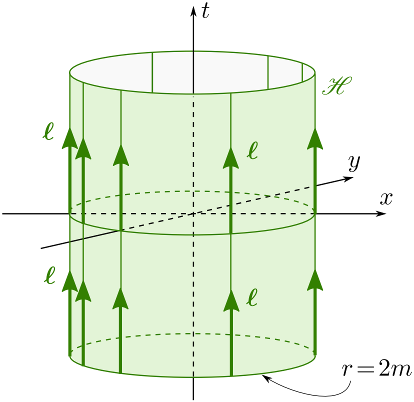
$$\Longrightarrow \ell^\alpha = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$$

$$\boxed{\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{x}{r}\boldsymbol{\partial}_x + \frac{y}{r}\boldsymbol{\partial}_y + \frac{z}{r}\boldsymbol{\partial}_z}$$

Example 3: Schwarzschild horizon

in Eddington-Finkelstein coordinates

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{4m}{r}\, dt\, dr + \left(1 + \frac{2m}{r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$



$$u := \left(1 - \frac{r}{2m}\right) \exp\left(\frac{r-t}{4m}\right) = 0$$

$$\mathscr{H} : \quad u = 0 \iff r = 2m$$

$$\boldsymbol{\nabla} u = \frac{1}{4m} e^{(r-t)/(4m)} \left[-\left(1 - \frac{r}{2m}\right) \mathbf{d}t - \left(1 + \frac{r}{2m}\right) \mathbf{d}r \right]$$

Exercise: compute $\boldsymbol{\ell}$ with ρ chosen so that $\ell^t = 1$ and get

$$\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{r - 2m}{r + 2m} \boldsymbol{\partial}_r \implies \boldsymbol{\ell} \overset{\mathscr{H}}{=} \boldsymbol{\partial}_t$$

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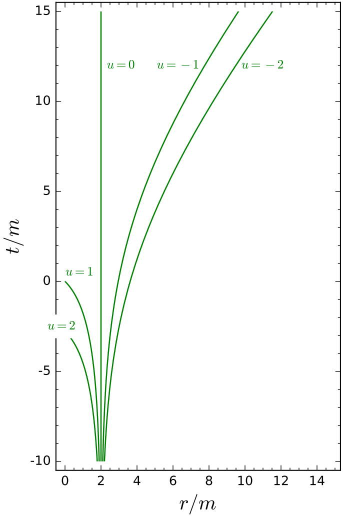

Hypersurfaces of constant value of u around the Schwarzschild horizon $u = 0$

Frobenius identity

A fundamental identity obeyed by any normal ℓ to a hypersurface

Starting point: $\boldsymbol{\ell} = -e^{\rho}\,\vec{\boldsymbol{\nabla}} u$

$$\Longrightarrow \quad \ell_\alpha = -e^{\rho}\,\nabla_\alpha u$$

$$\Longrightarrow \quad \nabla_\alpha \ell_\beta = -e^{\rho}\nabla_\alpha\rho\nabla_\beta u - e^{\rho}\nabla_\alpha\nabla_\beta u$$

$$\Longrightarrow \quad \nabla_\alpha \ell_\beta - \nabla_\beta \ell_\alpha = -e^{\rho}\nabla_\alpha\rho\nabla_\beta u + e^{\rho}\nabla_\beta\rho\nabla_\alpha u$$

$$\Longrightarrow \quad \boxed{\nabla_\alpha \ell_\beta - \nabla_\beta \ell_\alpha = \nabla_\alpha\rho\,\ell_\beta - \nabla_\beta\rho\,\ell_\alpha}$$

In terms of exterior (Cartan) calculus:

$$\boxed{\mathbf{d}\underline{\boldsymbol{\ell}} = \mathbf{d}\rho \wedge \underline{\boldsymbol{\ell}}}$$

where

- $\underline{\boldsymbol{\ell}}$ is the 1-form metric-dual to vector $\boldsymbol{\ell}$: $\underline{\boldsymbol{\ell}} = \ell_\alpha \mathbf{d}x^\alpha$, $\ell_\alpha = g_{\alpha\mu}\ell^\mu$
- $\mathbf{d}\underline{\boldsymbol{\ell}}$ is the exterior derivative of $\underline{\boldsymbol{\ell}}$ (2-form)
- $\wedge$ is the exterior product of p-forms

Null geodesic generators

Contract Frobenius identity with $\boldsymbol{\ell}$:

$$\ell^\mu \nabla_\mu \ell_\alpha - \ell^\mu \nabla_\alpha \ell_\mu = \ell^\mu \nabla_\mu \rho \, \ell_\alpha - \underbrace{\ell^\mu \ell_\mu}_{0} \nabla_\alpha \rho$$

Now $\ell^\mu \nabla_\alpha \ell_\mu = \nabla_\alpha \underbrace{(\ell^\mu \ell_\mu)}_{0} - \ell_\mu \nabla_\alpha \ell^\mu \implies \ell^\mu \nabla_\alpha \ell_\mu = 0$

Hence

$$\ell^\mu \nabla_\mu \ell_\alpha = \kappa \, \ell_\alpha \quad \text{with } \kappa := \ell^\mu \nabla_\mu \rho = \boldsymbol{\nabla}_{\boldsymbol{\ell}} \, \rho$$

or, by metric duality (index raising):

$$\ell^\mu \nabla_\mu \ell^\alpha = \kappa \, \ell^\alpha$$

i.e.

$$\boxed{\boldsymbol{\nabla}_{\boldsymbol{\ell}} \, \boldsymbol{\ell} = \kappa \, \boldsymbol{\ell}}$$

Null geodesic generators

$\boxed{\nabla_{\boldsymbol{\ell}}\, \boldsymbol{\ell} = \kappa\, \boldsymbol{\ell}}$ $\Longrightarrow$ $\boldsymbol{\ell}$ is a **pregeodesic vector**, i.e. $\exists$ rescaling factor α such that $\boldsymbol{\ell}' = \alpha \boldsymbol{\ell}$ is a **geodesic vector**: $\nabla_{\boldsymbol{\ell}'}\, \boldsymbol{\ell}' = 0$

Exercise: prove it!

$\Longrightarrow$ the field lines of $\boldsymbol{\ell}$ are (null) geodesics.

κ is called the **non-affinity coefficient** of the null normal $\boldsymbol{\ell}$ because

$$\kappa = 0 \iff \lambda \text{ is an affine parameter}$$

where λ is the parameter along a geodesic field line of $\boldsymbol{\ell}$ whose derivative vector is $\boldsymbol{\ell}$:

$$\boldsymbol{\ell} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\lambda}$$

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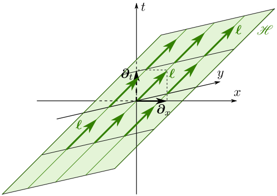
Any null hypersurface $\mathscr{H}$ is ruled by a family of null geodesics, called the **generators of** $\mathscr{H}$, and each vector field $\boldsymbol{\ell}$ normal to $\mathscr{H}$ is tangent to these null geodesics.

Examples of null geodesic generators

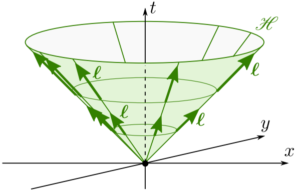
null hyperplane



$$\nabla_{\ell}\,\ell = 0$$

$$\kappa = 0$$

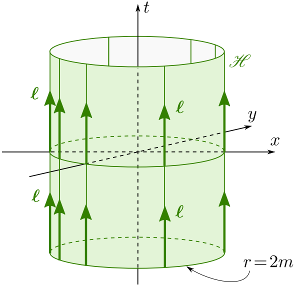
future null cone



$$\nabla_{\ell}\,\ell = 0$$

$$\kappa = 0$$

Schwarzschild horizon



$$\nabla_{\ell}\,\ell = \kappa\,\ell$$

$$\kappa = \frac{1}{4m}$$

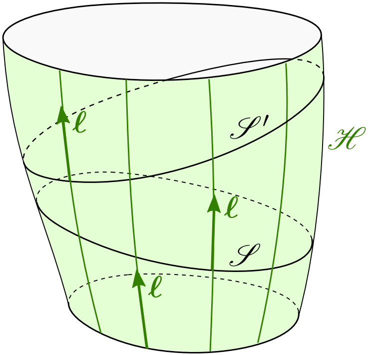
Using SageMath to compute κ for the Schwarzschild horizon

SageMath: Python-based free mathematics software system with tensor calculus capabilities (cf. https://sagemanifolds.obspm.fr)

The computation of κ:
https://nbviewer.jupyter.org/github/egourgoulhon/BHLectures/blob/master/sage/Schwarzschild_horizon.ipynb

See also
https://luth.obspm.fr/~luthier/gourgoulhon/bh16/sage.html
for all the notebooks associated with these lectures

Cross-sections of a null hypersurface



cross-section of the null hypersurface $\mathscr{H}$: 2-dimensional submanifold $\mathscr{S} \subset \mathscr{H}$ such that

1. the null normal $\boldsymbol{\ell}$ is nowhere tangent to $\mathscr{S}$
2. each null geodesic generator of $\mathscr{H}$ intersects $\mathscr{S}$ once, and only once

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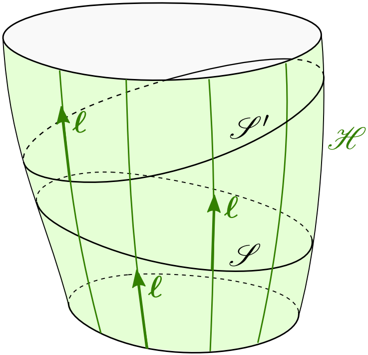

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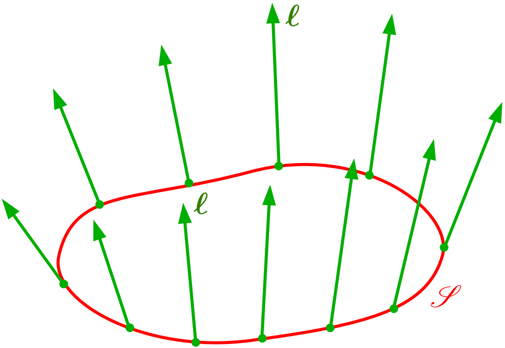
Any cross-section $\mathscr{S}$ is spacelike, i.e. all vectors tangent to $\mathscr{S}$ are spacelike.

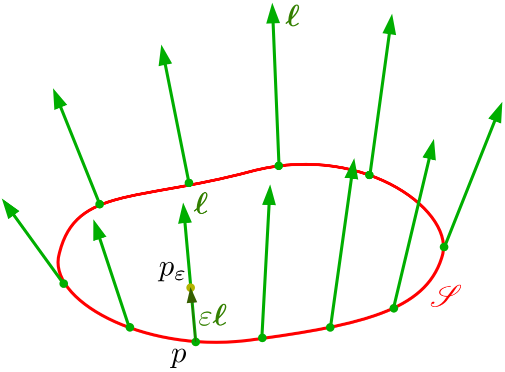
Proof: a vector tangent to $\mathscr{H}$ cannot be timelike, nor null and not normal.

Expansion along a null normal



1. Consider a cross-section $\mathscr{S}$ and a null normal $\boldsymbol{\ell}$ to $\mathscr{H}$

Expansion along a null normal



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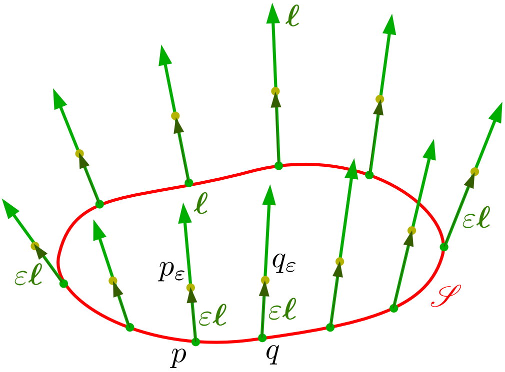
2. ε being a small parameter, displace the point p by the vector $\varepsilon\boldsymbol{\ell}$ to the point p_ε

Expansion along a null normal



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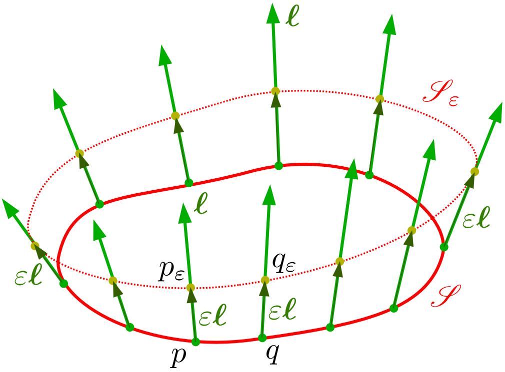

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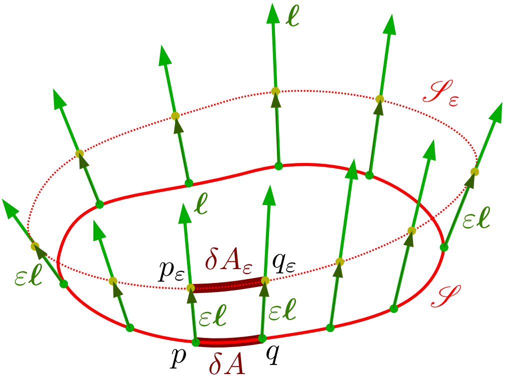

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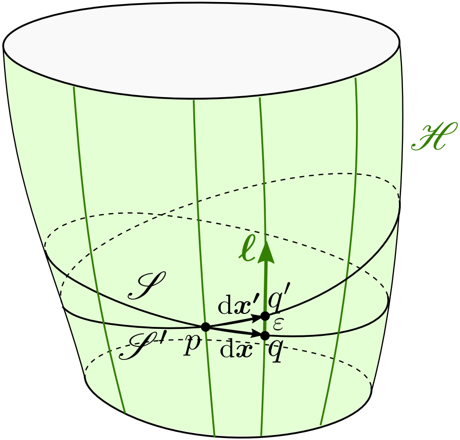
At each point, the **expansion along** $\boldsymbol{\ell}$ is defined from the relative change of the area element δA:

$$\theta_{(\boldsymbol{\ell})} := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \frac{\delta A_\varepsilon - \delta A}{\delta A} = \mathcal{L}_{\boldsymbol{\ell}} \ln\sqrt{q} = q^{\mu\nu}\nabla_\mu \ell_\nu$$

Expansion along a null normal



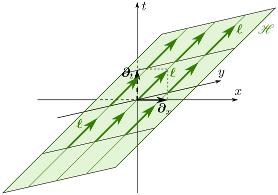
The expansion $\theta_{(\boldsymbol{\ell})}$ at a point $p \in \mathscr{H}$ depends solely on the null normal $\boldsymbol{\ell}$, not on the choice of the cross-section $\mathscr{S}$ through p.

Dependency of $\theta_{(\boldsymbol{\ell})}$ w.r.t. $\boldsymbol{\ell}$:

$$\boldsymbol{\ell}' = \alpha \boldsymbol{\ell} \Longrightarrow \theta_{(\boldsymbol{\ell}')} = \alpha \theta_{(\boldsymbol{\ell})}$$

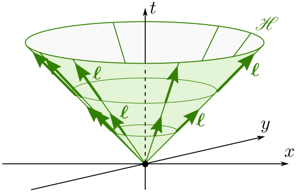
Examples of expansions

null hyperplane



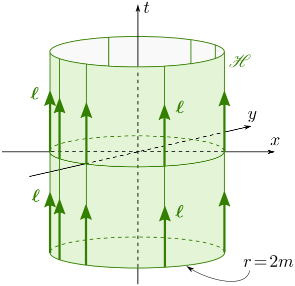
$$\theta_{(\boldsymbol{\ell})} = 0$$

future null cone



$$\theta_{(\boldsymbol{\ell})} = \frac{2}{r}$$

Schwarzschild horizon



$$\theta_{(\boldsymbol{\ell})} = 0$$

Outline

Distinguishing a black hole horizon from a generic null hypersurface

Recall the naive definition stated above:

A **black hole** is a localized region of spacetime from which neither massive particles nor massless ones (photons) can escape.

- no-escape facet $\Longrightarrow$ boundary = null hypersurface
  But we don't want the interior of a future null cone in Minkowski spacetime to be called a black hole...

Distinguishing a black hole horizon from a generic null hypersurface

Recall the naive definition stated above:

A **black hole** is a localized region of spacetime from which neither massive particles nor massless ones (photons) can escape.

- no-escape facet $\Longrightarrow$ boundary = null hypersurface
  But we don't want the interior of a future null cone in Minkowski spacetime to be called a black hole...
- localized facet: *for equilibrium configurations*, can be enforced by cross-sections = closed surfaces with *constant* area, i.e. vanishing expansion

Non-expanding horizons

Definition

A **non-expanding horizon (NEH)** is a null hypersurface $\mathscr{H}$ whose cross-sections $\mathscr{S}$ are *closed* surfaces (i.e. compact without boundary) and such that the expansion along any null normal $\boldsymbol{\ell}$ vanishes identically:

$$\boxed{\theta_{(\boldsymbol{\ell})} = 0}$$

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$$\theta_{(\boldsymbol{\ell})} = 0$$

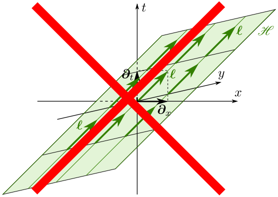
Remark 1: definition independent of ℓ, due to $\boldsymbol{\ell}' = \alpha\boldsymbol{\ell} \implies \theta_{(\boldsymbol{\ell}')} = \alpha\theta_{(\boldsymbol{\ell})}$

Remark 2: most of the time, the cross-sections $\mathscr{S}$ are assumed to have the $\mathbb{S}^2$ topology, so that $\mathscr{H}$ has the "cylinder" topology: $\mathscr{H} \simeq \mathbb{R} \times \mathbb{S}^2$.

Remark 3: concept introduced by P. Hájíček in 1973 [Com. Math. Phys. **34**, 37] under the name *perfect horizon*; the term *non-expanding horizon* has been coined by A. Ashtekar, S. Fairhurst & B. Krishnan in 2000 [PRD **62**, 104025].

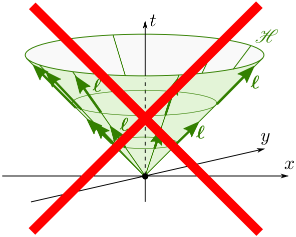
(Counter-)examples of non-expanding horizons

null hyperplane



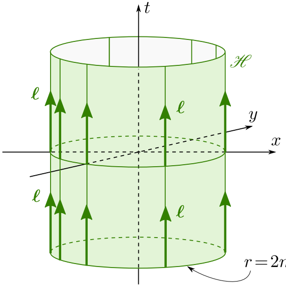
no closed cross-sections

future null cone



nonzero expansion

Schwarzschild horizon

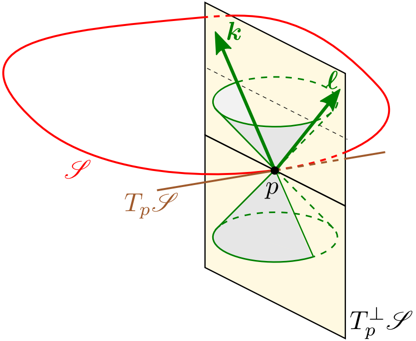


OK

Connection with marginally trapped surfaces

Definition of a trapped surface (1/2)

$\mathscr{S}$: closed (compact without boundary) spacelike 2-dimensional surface embedded in spacetime $(\mathscr{M}, \boldsymbol{g})$



Being spacelike, $\mathscr{S}$ lies outside the light cone $\Longrightarrow$ $\exists$ two future-directed null directions orthogonal to $\mathscr{S}$:
$\boldsymbol{\ell}$ = outgoing, expansion $\theta_{(\boldsymbol{\ell})}$
$\boldsymbol{k}$ = ingoing, expansion $\theta_{(\boldsymbol{k})}$

In Minkowski spacetime:
$\theta_{(\boldsymbol{k})} < 0$ and $\theta_{(\boldsymbol{\ell})} > 0$

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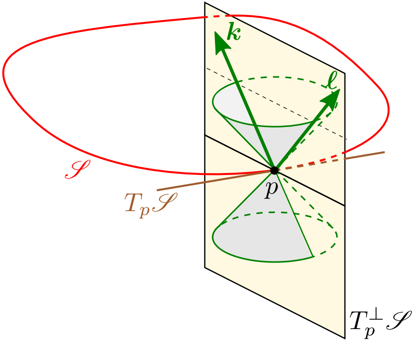

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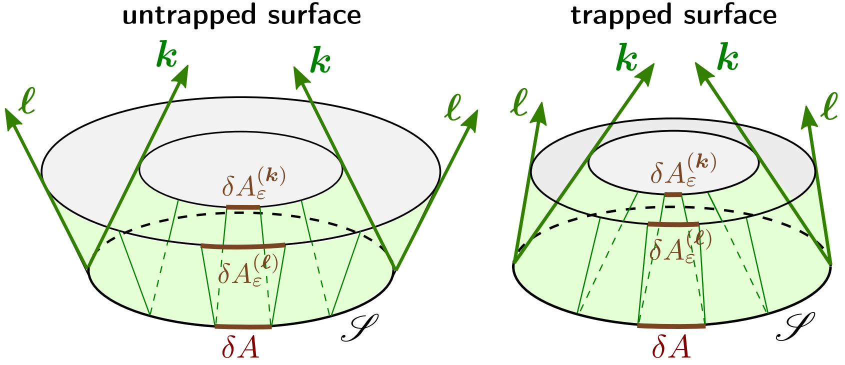
$\mathscr{S}$ is **trapped** $\iff$ $\theta_{(\boldsymbol{k})} < 0$ and $\theta_{(\boldsymbol{\ell})} < 0$ [Penrose 1965]

$\mathscr{S}$ is **marginally trapped** $\iff$ $\theta_{(\boldsymbol{k})} < 0$ and $\theta_{(\boldsymbol{\ell})} = 0$

Connection with marginally trapped surfaces

Definition of a trapped surface (2/2)



$\theta_{(\boldsymbol{k})} < 0$ and $\theta_{(\boldsymbol{\ell})} > 0$

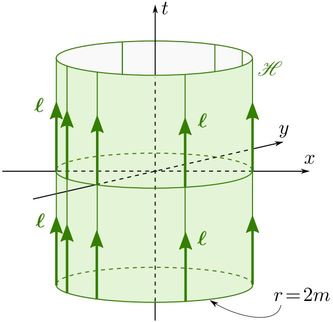
$\theta_{(\boldsymbol{k})} < 0$ and $\theta_{(\boldsymbol{\ell})} < 0$

No trapped surface in Minkowski spacetime
$\Longrightarrow$ *trapped surface* = **local** concept characterizing strong gravity (cf. Badri Krishnan's lecture)

Connection with marginally trapped surfaces

Generically, one has $\theta_{(\boldsymbol{k})} < 0$ along cross-sections of a non-expanding horizon. Hence:

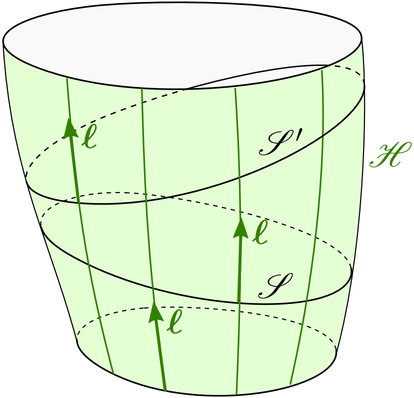
A non-expanding horizon is (generically) a null hypersurface foliated by marginally trapped surfaces.



Example: Schwarzschild horizon

$$\theta_{(\boldsymbol{k})} = -\frac{1}{m} \quad \text{and} \quad \theta_{(\boldsymbol{\ell})} = 0$$

Area of a non-expanding horizon



Each cross-section $\mathscr{S}$ of $\mathscr{H}$ is a *spacelike* closed surface.

The area of $\mathscr{S}$ is given by the positive definite metric $\boldsymbol{q}$ induced by $\boldsymbol{g}$ on $\mathscr{S}$:

$$A = \int_{\mathscr{S}} \sqrt{q}\, \mathrm{d}y^1 \mathrm{d}y^2$$

where $y^a = (y^1, y^2)$ are coordinates on $\mathscr{S}$ and $q := \det(q_{ab})$

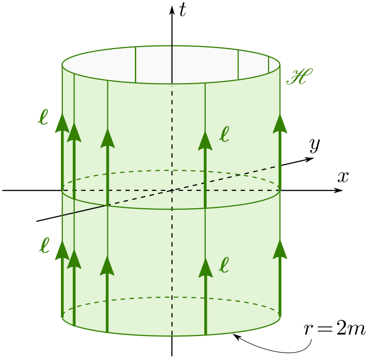
Since $\theta_{(\boldsymbol{\ell})} = 0$, we have:

On a non-expanding horizon, the area A is independent of the choice of the cross-section $\mathscr{S}$ $\Longrightarrow$ **area of** $\mathscr{H}$

Example: area of the Schwarzschild horizon

Spacetime metric:

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \frac{4m}{r}\,\mathrm{d}t\,\mathrm{d}r + \left(1+\frac{2m}{r}\right)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2$$



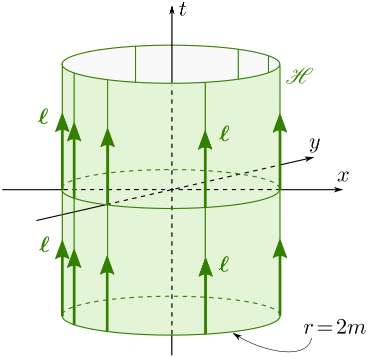
$\mathscr{H}$: $r = 2m$; coord: (t, θ, φ)

$\mathscr{S}$: $r = 2m$ and $t = t_0$; coord: $y^a = (\theta, \varphi)$

Example: area of the Schwarzschild horizon

Spacetime metric:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{4m}{r}\, dt\, dr + \left(1 + \frac{2m}{r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$



$\mathscr{H}$: $r = 2m$; coord: (t, θ, φ)

$\mathscr{S}$: $r = 2m$ and $t = t_0$; coord: $y^a = (\theta, \varphi)$

$\Longrightarrow$ induced metric on $\mathscr{S}$:

$$q_{ab}\, dy^a dy^b = (2m)^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$\Longrightarrow q := \det(q_{ab}) = (2m)^4 \sin^2\theta$$

$$\Longrightarrow A = \int_{\mathscr{S}} (2m)^2 \sin\theta\, d\theta d\varphi$$

$$\Longrightarrow \boxed{A = 16\pi m^2}$$

An important subclass of NEH: Killing horizons

Killing vector: generator $\boldsymbol{\xi}$ of a 1-parameter symmetry group of the spacetime $(\mathscr{M}, \boldsymbol{g})$ (isometries)
$(\mathscr{M}, \boldsymbol{g})$ is invariant along the field lines of $\boldsymbol{\xi}$:

$$\mathcal{L}_{\boldsymbol{\xi}}\, \boldsymbol{g} = 0 \iff \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$$

Definition

A **Killing horizon** is a null hypersurface $\mathscr{H}$ in a spacetime $(\mathscr{M}, \boldsymbol{g})$ endowed with a Killing vector field $\boldsymbol{\xi}$ such that, on $\mathscr{H}$, $\boldsymbol{\xi}$ is normal to $\mathscr{H}$.

$\Longrightarrow$ $\boldsymbol{\xi}$ is null on $\mathscr{H}$

$\Longrightarrow$ the null geodesic generators of $\mathscr{H}$ are orbits of the 1-parameter group of isometries generated by $\boldsymbol{\xi}$.

Killing horizons as non-expanding horizons

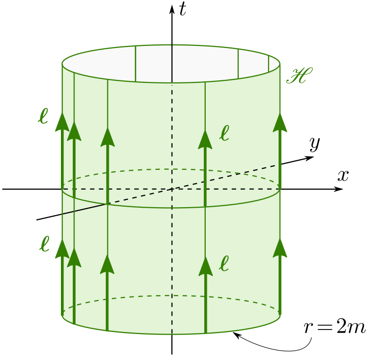
A Killing horizon with closed cross-sections is a non-expanding horizon.

Proof: since $\boldsymbol{\xi}$ is a symmetry generator and $\boldsymbol{\xi} = \boldsymbol{\ell}$ on $\mathscr{H}$, the area δA of an element of cross-section does not vary when Lie-dragged along $\boldsymbol{\ell}$, hence $\theta_{(\boldsymbol{\ell})} = 0$.

Example of Killing horizon: the Schwarzschild horizon

Spacetime metric:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \frac{4m}{r}\,\mathrm{d}t\,\mathrm{d}r + \left(1 + \frac{2m}{r}\right)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2$$



Killing vector field of Schwarzschild spacetime associated with stationarity: $\boldsymbol{\xi} = \boldsymbol{\partial}_t$

On $\mathscr{H}$: $\boldsymbol{\xi} = \boldsymbol{\ell}$

Outline

Limitation of the concept of non-expanding horizon

Non-expanding horizons capture well the "localized-in-space" feature of the black hole region. However they do so only for *steady-state configurations*, for which the cross section area remains constant. In particular, Killing horizons assume that the spacetime is endowed with some symmetry, usually *stationarity*.

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To define black holes in *dynamical* spacetimes, one shall define properly some "infinitely far" region and distinguish among the timelike or null worldlines, those that can reach the far region from those that cannot.

The definition of the "infinitely far" region is best performed via Penrose's concept of conformal completion, also called conformal compactification.

Conformal completion of Minkowski spacetime

1. Introducing "compactified" coordinates

Spacetime metric:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

Move to spherical coordinates (t, r, θ, φ):
$x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

$$\Longrightarrow \quad ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2$$

Move to coordinates $(\tau, \chi, \theta, \varphi)$ with $0 \leq \chi < \pi$ and $\chi - \pi < \tau < \pi - \chi$:

$$\begin{cases} \tau = \arctan(t+r) + \arctan(t-r) \\ \chi = \arctan(t+r) - \arctan(t-r) \end{cases} \iff \begin{cases} t = \dfrac{\sin\tau}{\cos\tau + \cos\chi} \\ r = \dfrac{\sin\chi}{\cos\tau + \cos\chi} \end{cases}$$

$$\Longrightarrow \quad ds^2 = (\cos\tau + \cos\chi)^{-2}\left[-d\tau^2 + d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

Conformal completion of Minkowski spacetime

2. The conformal metric

Thus we may write $\boldsymbol{g} = \Omega^{-2}\tilde{\boldsymbol{g}}$, or equivalently

$$\boxed{\tilde{\boldsymbol{g}} = \Omega^2 \boldsymbol{g}}$$

with

- $\Omega := \cos\tau + \cos\chi = \dfrac{2}{\sqrt{(t-r)^2+1}\sqrt{(t+r)^2+1}}$
- $\tilde{\boldsymbol{g}}$ is the metric defined by

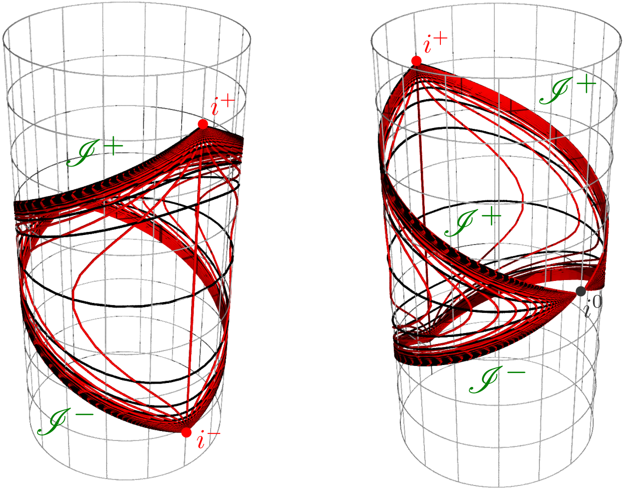
$$d\tilde{s}^2 = -d\tau^2 + \underbrace{d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)}_{\text{standard metric on } \mathbb{S}^3}$$

$\tilde{\boldsymbol{g}}$ is a Lorentzian metric on the **Einstein cylinder** $\mathscr{E} = \mathbb{R}\times\mathbb{S}^3$

$(\mathscr{E}, \tilde{\boldsymbol{g}})$ is a solution of Einstein's equation with a cosmological constant $\Lambda > 0$ and some pressureless matter of uniform density $\rho = \Lambda/(4\pi)$.

Conformal completion of Minkowski spacetime

3. Embedding into the Einstein cylinder

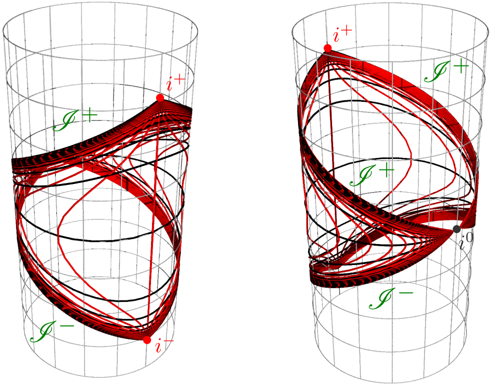


- on $\mathscr{E}$:
  $-\infty < \tau < +\infty$
  $0 \leq \chi \leq \pi$
- on $\mathscr{M}$:
  $\chi - \pi < \tau < \pi - \chi$
  $0 \leq \chi < \pi$

cf. https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/conformal_Minkowski.ipynb for an interactive 3D view

Conformal completion of Minkowski spacetime

3. Embedding into the Einstein cylinder



Boundaries of the embedding of $\mathscr{M}$ into $\mathscr{E}$:

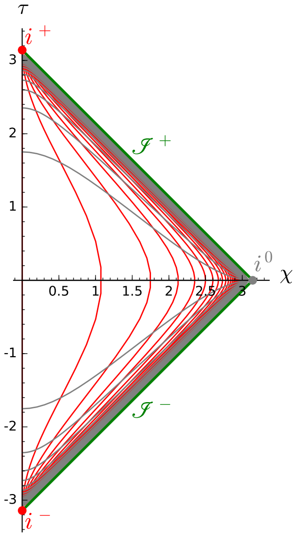
- $\mathscr{I}^+$ = hypersurface $\{\tau = \pi - \chi, \ 0 < \tau < \pi\}$
- $\mathscr{I}^-$ = hypersurface $\{\tau = \chi - \pi, \ -\pi < \tau < 0\}$
- i^0 = point $(\tau, \chi) = (0, \pi)$
- i^+ = point $(\tau, \chi) = (\pi, 0)$
- i^- = point $(\tau, \chi) = (-\pi, 0)$

Closure of $\mathscr{M}$ in $\mathscr{E}$: $\overline{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}^+ \cup \mathscr{I}^- \cup \{i^0, i^+, i^-\}$

NB: $\mathscr{I}^+$ and $\mathscr{I}^-$ are *not* parts of $\mathscr{M}$ and i^0, i^+ and i^- are *not* points of $\mathscr{M}$

Conformal completion of Minkowski spacetime

4. Conformal diagram

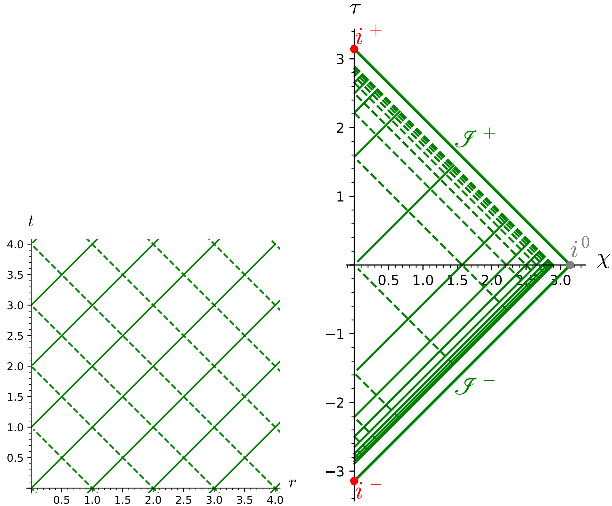


red: $r = \text{const}$

grey: $t = \text{const}$

Conformal completion of Minkowski spacetime

4. Conformal diagram

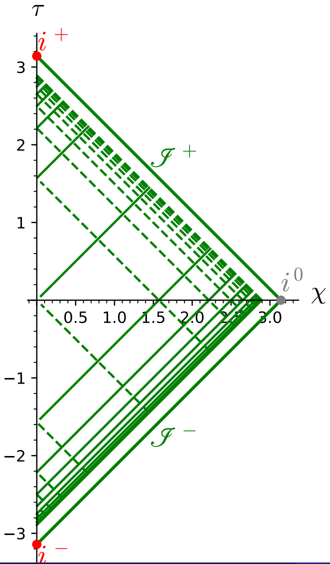


solid:
$u := t - r = \text{const}$
dashed:
$v := t + r = \text{const}$

Radial null geodesics appear as straight lines with $\pm 45°$ slope (conformal diagram)

Conformal completion of Minkowski spacetime

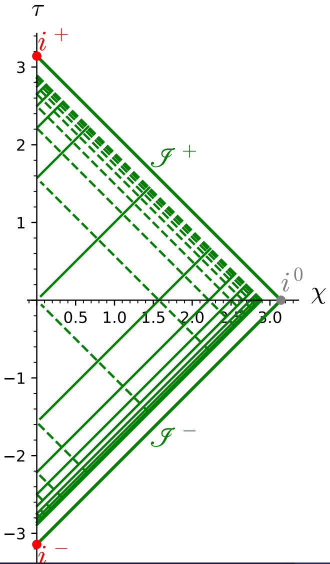
4. Conformal diagram



- $\mathscr{I}^+$: where all radial future-directed null geodesics terminate $\Longrightarrow$ **future null infinity**
- $\mathscr{I}^-$: where all radial future-directed null geodesics originate $\Longrightarrow$ **past null infinity**

Conformal completion of Minkowski spacetime

4. Conformal diagram



- $\mathscr{I}^+$: where all radial future-directed null geodesics terminate $\Longrightarrow$ **future null infinity**
- $\mathscr{I}^-$: where all radial future-directed null geodesics originate $\Longrightarrow$ **past null infinity**

Let $\mathscr{I} := \mathscr{I}^+ \cup \mathscr{I}^-$ and $\tilde{\mathscr{M}} := \mathscr{M} \cup \mathscr{I}$

$\tilde{\mathscr{M}}$ is a manifold with boundary, and its boundary is $\mathscr{I}$. The conformal factor Ω relating $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$ vanishes at the boundary:

$$\boxed{\Omega \stackrel{\mathscr{I}}{=} 0}$$

$\boldsymbol{g} = \Omega^{-2}\tilde{\boldsymbol{g}} \Longrightarrow \mathscr{I}$ is "infinitely far" from any point in $\mathscr{M}$

Conformal completion

Definition 1

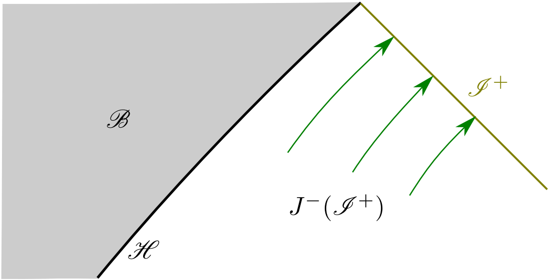
A spacetime $(\mathscr{M}, \boldsymbol{g})$ admits a **conformal completion** iff there exists a Lorentzian manifold with boundary $(\tilde{\mathscr{M}}, \tilde{\boldsymbol{g}})$ equipped with a smooth non-negative scalar field $\Omega : \tilde{\mathscr{M}} \to \mathbb{R}^+$ such that

- $\tilde{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}$, with $\mathscr{I} := \partial\tilde{\mathscr{M}}$ (the boundary of $\tilde{\mathscr{M}}$);
- on $\mathscr{M}$, $\tilde{\boldsymbol{g}} = \Omega^2 \boldsymbol{g}$;
- on $\mathscr{I}$, $\Omega = 0$;
- on $\mathscr{I}$, $\mathbf{d}\Omega \neq 0$.

Definition 2

$(\tilde{\mathscr{M}}, \tilde{\boldsymbol{g}})$ is a **conformal completion at null infinity** of $(\mathscr{M}, \boldsymbol{g})$ iff the boundary $\mathscr{I} := \partial\tilde{\mathscr{M}}$ obeys $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$, with $\mathscr{I}^+$ (resp. $\mathscr{I}^-$) being never intersected by any past-directed (resp. future-directed) causal curve originating in $\mathscr{M}$. $\mathscr{I}^+$ is called the **future null infinity** and $\mathscr{I}^-$ the **past null infinity** of $(\mathscr{M}, \boldsymbol{g})$.

General definition of a black hole, at last!



Causal past $J^-(\mathscr{I}^+)$: set of points of $\tilde{\mathscr{M}}$ that can be reached from a point of $\mathscr{I}^+$ by a past-directed causal (i.e. null or timelike) curve.

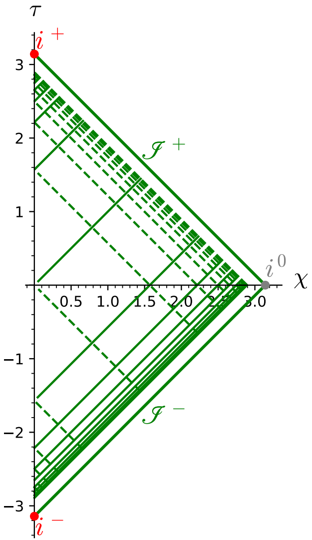
Definition

Let $(\mathscr{M}, \boldsymbol{g})$ be a spacetime with a conformal completion at null infinity such that $\mathscr{I}^+$ is complete; the **black hole region**, or simply **black hole**, is the set of points of $\mathscr{M}$ that are not in the causal past of the future null infinity:

$$\mathscr{B} := \mathscr{M} \setminus \left(J^-(\mathscr{I}^+) \cap \mathscr{M}\right)$$

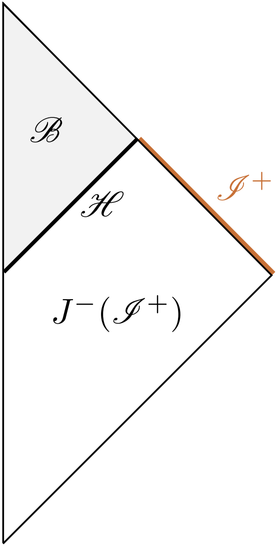
The boundary of $\mathscr{B}$ is called the **(future) event horizon**: $\mathscr{H} = \partial\mathscr{B}$

No black hole in Minkowski spacetime



$$J^-(\mathscr{I}^+) \cap \mathscr{M} = \mathscr{M} \Longrightarrow \mathscr{B} = \varnothing$$

Completeness of $\mathscr{I}^+$ to avoid spurious BH



If $\mathscr{I}^+$ is a null hypersurface, $\mathscr{I}^+$ complete $\iff$ $\mathscr{I}^+$ generated by complete null geodesics.

$\leftarrow$ Spurious black hole region $\mathscr{B}$ in Minkowski spacetime resulting from a conformal completion with a non-complete $\mathscr{I}^+$.

Properties of the event horizon of a black hole

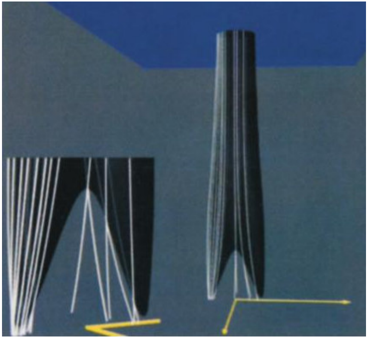
Property 1

The event horizon $\mathscr{H}$ is an **achronal set**, i.e. no pair of points of $\mathscr{H}$ can be connected by a timelike curve of $\mathscr{M}$.

Property 2

$\mathscr{H}$ is a topological manifold of dimension 3.

Properties of the event horizon



[R.A. Matzner et al., Science **270**, 941 (1995)]

Property 3 (Penrose 1968)

$\mathscr{H}$ is ruled by a family of *null geodesics* that

- either lie entirely in $\mathscr{H}$ or never leave $\mathscr{H}$ when followed into the future from the point where they arrived in $\mathscr{H}$
- have no endpoint in the future.

Moreover, there is exactly one null geodesic through each point of $\mathscr{H}$, except at special points where null geodesics enter in contact with $\mathscr{H}$.

Property 4

Wherever it is smooth, $\mathscr{H}$ is a null hypersurface.