Gravitational-wave tails-of-memory

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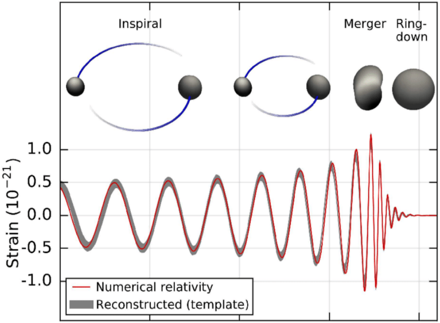
Goal: phase and gravitational waveform at 4PN in GR

Trestini, Larrouturou, Blanchet (arXiv:2209.02719)
Trestini, Blanchet (arXiv: 2210.xxxxx)

Motivation

Why such high order ?

- next-generation detectors (ET, CE, LISA) need ∼5PN waveforms
- numerical relativity calibrate their initial data with the PN expansion
- improving effective-one-body models
- increase accuracy of astrophysical simulations (see talk by Sobolenko)



The linearized vacuum solution

In order to solve the full non-linear equations in the vacuum zone up to some PN order, the first step is to solve the linearized solution, i.e. solve :

$$\Box_\eta h_1^{\mu\nu} = 0$$
$$\partial_\nu h_1^{\mu\nu} = 0$$

The most general solution can be parametrized by a set of multipolar moments M_L and S_L (resp. mass- and current-type canonical multipoles). Schematically, we have :

$$h_1 \sim \sum_{\ell=0}^{\infty} \left(\partial_L \left[\frac{1}{r} M_L(u) \right] + \varepsilon_{abc} \partial_L \left[\frac{1}{r} S_L(u) \right] \right)$$

These can be matched to the solution in the near zone. The source multipoles necessary for the 4PN waveform have already been computed (e.g. Larrouturou+ '21).

Full vacuum PM solution

Once the multipolar moments determined, the full solution is determined by the post-Minkowskian algorithm (Blanchet & Damour '85). Write the metric as a formal PM expansion:

$$h^{\mu\nu} = G h_1^{\mu\nu} + G^2 h_2^{\mu\nu} + G^3 h_3^{\mu\nu} + \dots$$

At each order $n \geq 2$, we need to solve

$$\Box h_n^{\mu\nu} = \Lambda_n^{\mu\nu}[h_1, \dots, h_{n-1}] \qquad \text{and} \qquad \partial_\nu h_n^{\mu\nu} = 0$$

This can then be integrated using the finite part procedure and harmonicity algorithm (Blanchet '16, LRR):

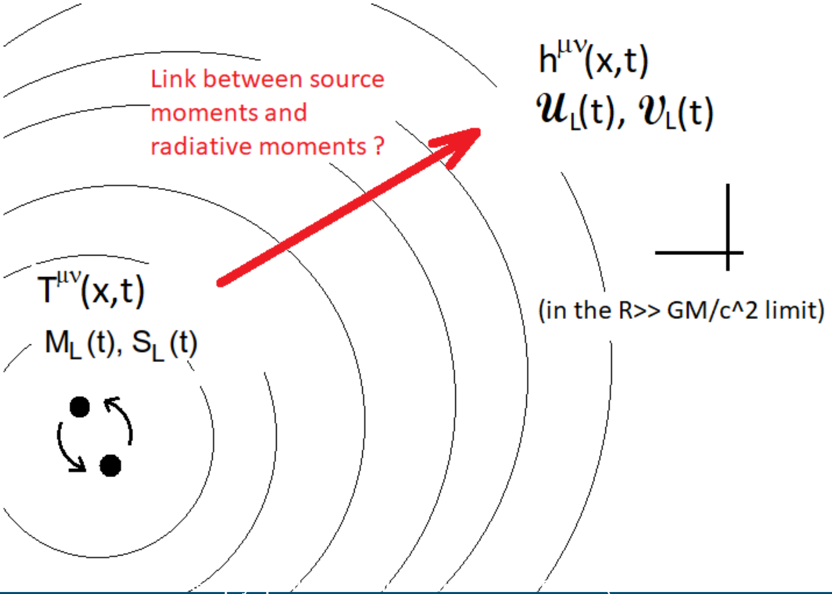
$$h_n^{\mu\nu} = u_n^{\mu\nu} + v_n^{\mu\nu}$$

where

$$u_n^{\mu\nu} = \underset{B=0}{\mathrm{FP}} \Box^{-1} \left[\left(\frac{r}{r_0} \right)^B \Lambda_n^{\mu\nu} \right]$$

$$v_n^{\mu\nu} = \mathcal{V}^{\mu\nu}[u_n] \quad \text{such that} \quad \Box v_n^{\mu\nu} = 0 \quad \text{and} \quad \partial_\nu v_n^{\mu\nu} = -\partial_\nu u^{\mu\nu}$$

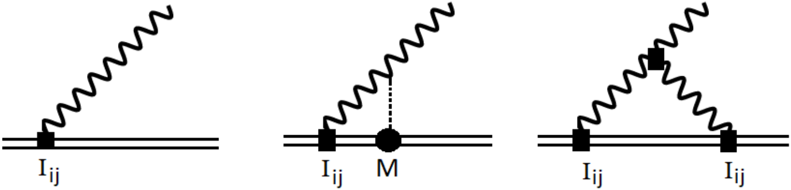
Source and radiative moments



Non linear effects

At 4.5PN, the radiative quadrupole is related to the source quadrupole by:

$$\begin{aligned}\mathcal{U}_{ij} = \overset{(2)}{\mathrm{M}}_{ij} &+ G\Big\{\mathrm{Tail}[\mathrm{M}\times M_{ij}] + \mathrm{Memory}[\mathrm{M}_{ab}\times \mathrm{M}_{ij}]\Big\} \\ &+ G^2\Big\{\text{Tail-of-tail}[\mathrm{M}\times\mathrm{M}\times\mathrm{M}_{ij}] + [\mathrm{M}\times \mathrm{S}_k \times \mathrm{M}_{ij}] \\ &\qquad + \text{Tail-of-memory}[\mathrm{M}\times\mathrm{M}_{ij}\times\mathrm{M}_{pq}]\Big\} \\ &+ G^3\Big\{\mathrm{Tail}^3[\mathrm{M}\times\mathrm{M}\times\mathrm{M}\times\mathrm{M}_{ij}]\Big\}\end{aligned}$$



Linear quadrupolar wave — Tail — Memory effect

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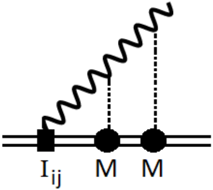
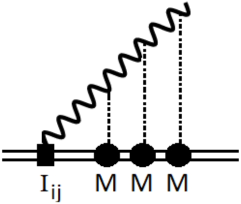
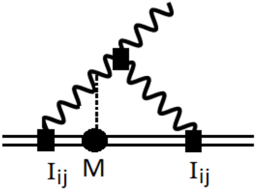


Tails-of-tails



Tails-of-tails-of-tails



Tails-of-memory

The quadratic tail

At quadratic order, the MPM construction yields non-local contributions to the radiative moment, called *tails*. The equation we need to solve is

$$\Box h_2^{\mu\nu} = N^{\mu\nu}[h_1, h_1] \qquad \text{and} \qquad \partial_\nu h_2^{\mu\nu} = 0$$

where $N^{\mu\nu}$ is a well-known functional that is quadratic in its arguments. The solution can be written

$$h_2 \sim M \int_1^\infty \mathrm{d}x \, Q_m(x) F(t - rx) + \dots, \quad \text{where } F \text{ is related to } \mathrm{M}_{ij}^{(4)}$$

where $Q_m(x)$ is the Legendre function of second type defined as

$$Q_\ell(x) = \frac{1}{2} P_m(x) \ln\left(\frac{x+1}{x-1}\right) - \sum_{j=1}^{\ell} \frac{1}{j} P_{\ell-j}(x) P_{j-1}(x)$$

This reduced in the $r \to \infty$ limit to the well-known expression

$$h_2 \sim M \int_0^\infty \mathrm{d}\tau \, \ln\left(\frac{\tau}{2b_0}\right) F(t - r - \tau) + \dots$$

Equation to solve

At cubic order, we inject the quadratic tail metric into the cubic source, and need to solve

$$\Box h_3^{\mu\nu} = 2N^{\mu\nu}[h_1, h_2] + M^{\mu\nu}[h_1, h_1, h_1] \qquad \text{and} \qquad \partial_\nu h_n^{\mu\nu} = 0$$

The difficult master equation we now have to solve is

$$\boxed{\Box_{k,m}\Psi_\ell = \frac{\hat{n}_L}{r^k} G(t-r) \int_1^{\infty} \mathrm{d}x\, Q_m(x) F(t - rx)}$$

where $G(u) \propto \overset{(p)}{\mathrm{M}}_{ij}(u)$, $F(u) \propto \overset{(q)}{\mathrm{M}}_{ij}(u)$, $\hat{n}_L \equiv \mathrm{STF}_{i_1 \dots i_\ell}\left[n_{i_1} \dots n_{i_\ell}\right]$

Formal solution

A formal solution of the previous equation is given by

$$
\begin{aligned}
{}_{k,m}\Psi_L &= \underset{B=0}{\mathrm{FP}}\,\Box^{-1}\left[\frac{\hat{n}_L}{r^k}\left(\frac{r}{r_0}\right)^B G(t-r)\int_1^\infty \mathrm{d}x Q_m(x)F(t-rx)\right] \\
&= \underset{B=0}{\mathrm{FP}}\int_\infty^{t-r}\mathrm{d}s\,\hat{\partial}_L\left[\frac{\mathcal{R}_\infty^B\left(\frac{t-r-s}{2},s\right)-\mathcal{R}_\infty^B\left(\frac{t+r-s}{2},s\right)}{r}\right]
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{R}_\infty^B(\rho,s) = -\rho^\ell G(s)\int_\rho^\infty \mathrm{d}\lambda\left(\frac{\lambda}{r_0}\right)^B\lambda^{-k}\frac{(\rho-\lambda)^\ell}{\ell!}\left(\frac{2}{\lambda}\right)^{\ell-1} \\
\times\int_1^\infty \mathrm{d}x\,Q_m(x)F(s-\lambda(x-1))
\end{aligned}
$$

This is *not* satisfactory: we need to remove explicitly the finite part prescription

Explicit solution for $k \in \{1, 2\}$

The solution to

$$\Box_{k,m} \Psi_\ell = \frac{\hat{n}_L}{r^k} G(t-r) \int_1^\infty \mathrm{d}x \, Q_m(x) F(t - rx)$$

is given by

$$\boxed{{}_{k,m}\Psi_L = -\frac{\hat{n}_L}{2r} \int_0^\infty \mathrm{d}\rho \, G(u-\rho) \int_0^\infty \mathrm{d}\tau \, F(u-\rho-\tau) \, {}_{k,m}K_\ell(\rho,\tau,r)}$$

where the kernel has a structure

$$K_\ell(\rho,\tau,r) \sim \tau^{1-k} \sum_{\substack{j \in \mathbb{Z} \\ p,q,n,s \in \mathbb{N}}} \left(\frac{\tau}{\rho}\right)^j \ln^s\left(\frac{r}{r_0}\right) \ln^p\left(\frac{\tau}{2r_0}\right) \ln^q\left(\frac{\rho}{2r_0}\right) \mathrm{Li}_n\left(-\frac{\tau}{\rho}\right)$$

Obtaining the solution for $k \geq 3$

For $k \geq 3$, integrate source by parts: decrease are powers of $1/r$ down to the cases $k = 1$ and $k = 2$ using the equation:

$$\begin{aligned}
&\hat{n}_L r^{B-k} G(t-r) \int_1^\infty \mathrm{d}x Q_m(x) F(t-rx) \\
&= \Box \left[\frac{\hat{n}_L r^{B-k+2}}{(k+\ell-2-B)(k-\ell-3-B)} G(t-r) \int_1^\infty \mathrm{d}x\, Q_m(x) F(t-rx) \right] \\
&\quad - \frac{2(k-3-B)\hat{n}_L r^{B-k+1}}{(k+\ell-2-B)(k-\ell-3-B)} \Bigg(\overset{(1)}{G}(t-r) \int_1^\infty \mathrm{d}x\, Q_m(x) F(t-rx) \\
&\qquad + G(t-r) \int_1^\infty \mathrm{d}x\, x\, Q_m(x) \overset{(1)}{F}(t-rx) \Bigg) \\
&\quad - \frac{\hat{n}_L r^{B-k+2}}{(k+\ell-2-B)(k-\ell-3-B)} \Bigg(2 \overset{(1)}{G}(t-r) \int_1^\infty \mathrm{d}x\, (x-1) Q_m(x) \overset{(1)}{F}(t-rx) \\
&\qquad + G(t-r) \int_1^\infty \mathrm{d}x\, (x^2-1)\, Q_m(x) \overset{(2)}{F}(t-rx) \Bigg)
\end{aligned}$$

and then use $x\, Q_m(x) = \frac{m+1}{2m+1} Q_{m+1}(x) + \frac{m}{2m+1} Q_{m-1}(x)$ if $m \geq 1$, and $x\, Q_1(x) = Q_0(x) + 1$.

Poles

When implementing the recursion relation, we find simple poles (i.e. $1/B$) for certain values of k and ℓ. This means that for $k \in \{1, 2\}$, we also have to compute

$$_{k,m}\chi_L = \underset{B=0}{\mathrm{FP}}\,\Box^{-1}\left[\frac{1}{B}\frac{\hat{n}_L}{r^k}\left(\frac{r}{r_0}\right)^B G(t-r)\int_1^\infty \mathrm{d}x\, Q_m(x) F(t-rx)\right]$$

Double poles (i.e. $1/B^2$) do not appear !

Removing logarithms

With the previous construction the asymptotic structure of waveform is:

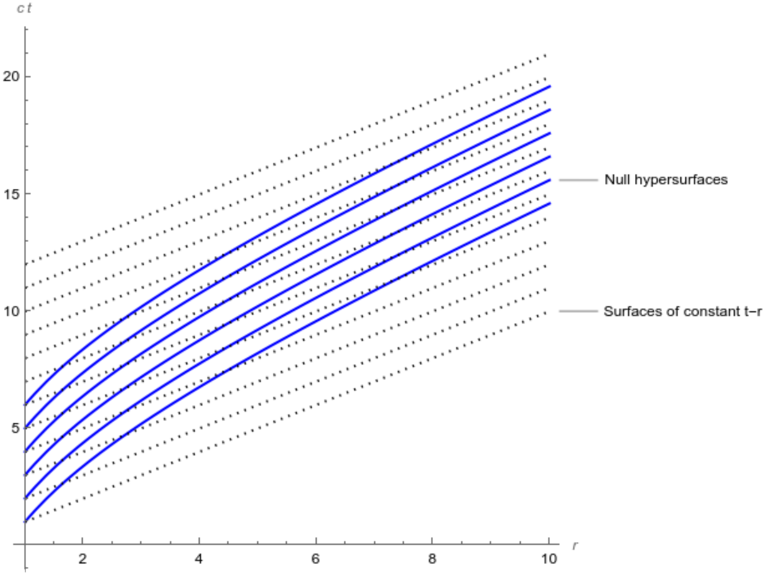
$$h_n \sim \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{p=0}^{\infty} \log^p(r) \hat{n}_L F_L(t-r) + \mathcal{O}\left(\frac{1}{r^{2-\epsilon}}\right)$$

Logarithms spoil the multipolar structure of the $1/r$ piece of the metric (cannot define the observable radiative moments $\mathcal{U}_{ij}$ and $\mathcal{V}_{ij}$) .

Coordinate singularity of the harmonic gauge: can be removed! In previous work (Blanchet & Damour '92, Blanchet '05, Marchand+ '16), a non-linear gauge transformation $x'^{\mu} = \varphi(x^{\mu})$ was guessed.

But for $\mathrm{M} \times \mathrm{M}_{ij} \times \mathrm{M}_{ab}$, no coordinate transformation was guessed: we have to remove them at each order via the **radiative algorithm**.

Retarded time: not a null coordinate!



In **Trestini, Larrouturou, Blanchet 2022** (arXiv: 2209.02719), we construct the metric in the radiative algorithm pioneered in Blanchet 1987. We find that the harmonic and radiative metrics are related by a coordinate transformation and a canonical moment redefinition:

$$\overline{\mathrm{M}}_{ij} = \mathrm{M}_{ij} - \frac{26}{15}\frac{G\,\mathrm{M}}{c^3}\,\mathrm{M}_{ij}^{(1)} + \frac{124}{45}\frac{G^2\mathrm{M}^2}{c^6}\,\mathrm{M}_{ij}^{(2)} + \frac{G^2\mathrm{M}}{c^8}\left[-\frac{8}{21}\,\mathrm{M}_{a\langle i}\mathrm{M}_{j\rangle a}^{(4)} - \frac{8}{7}\,\mathrm{M}_{a\langle i}^{(1)}\mathrm{M}_{j\rangle a}^{(3)} - \frac{8}{9}\,\varepsilon_{ab\langle i}\mathrm{M}_{j\rangle a}^{(3)}\mathrm{S}_b\right],$$

We successfully remove the logarithms from the full metric, so in particular from the $1/r$ piece

$$h_n^{\mathrm{rad}} \sim \frac{1}{r}\sum_{\ell=0}^{\infty}\hat{n}_L F_L(t-r) + \mathcal{O}\left(\frac{1}{r^2}\right)$$

This asymptotic metric has a multipolar structure and we can read off the radiative moments $\mathcal{U}_L$ and $\mathcal{V}_L$.

Simplification

Although this method provides an explicit solution, it is very complex. To simplify it, we tried to express the end result with one big kernel using integration by parts

$$\mathcal{U}_{ij} = \mathrm{M}\int_{\epsilon}^{+\infty} \mathrm{d}\rho\, \overline{\mathrm{M}}_{a\langle i}(u-\rho) \int_{0}^{+\infty} \mathrm{d}\tau\, \overline{\mathrm{M}}^{(8)}_{j\rangle a}(u-\rho-\tau)\, \Omega(\rho,\tau) + \mathcal{S}_{\epsilon}$$

where ϵ is a regularization which should cancel out in the end result when $\epsilon \to 0$. Being agnostic, one would expect Ω to be as complect as the kernel K, and involve polylogarithms ... but we instead find that:

$$\Omega(\rho,\tau) = \frac{7613764}{165375} - \frac{1024076}{18375}\frac{\tau}{\rho} - \frac{2074}{63}\left(\frac{\tau}{\rho}\right)^2 - \frac{104}{15}\left(\frac{\tau}{\rho}\right)^3 + \frac{634076}{55125}\ln\left(\frac{\rho}{2r_0}\right) + \frac{384}{175}\frac{\tau}{\rho}\ln\left(\frac{\rho}{2r_0}\right) - \frac{144}{175}\ln\left(\frac{\rho}{2r_0}\right)^2 + \frac{8}{7}\ln\left(\frac{\tau}{2r_0}\right)$$

Result

$$
\begin{aligned}
\mathcal{U}_{ij}^{\mathrm{M}\times\mathrm{M}_{ij}\times\mathrm{M}_{ij}} = \frac{2G^2\mathrm{M}}{7c^8}\Bigg\{ & c_1\int_0^{+\infty}\mathrm{d}\rho\,\mathrm{M}^{(4)}_{a\langle i}(u-\rho)\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(4)}_{j\rangle a}(u-\rho-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+c_2\right] \\
&+\int_0^{+\infty}\mathrm{d}\tau\,(\mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(4)}_{j\rangle a})(u-\tau)\left[c_3\ln\left(\frac{\tau}{2b_0}\right)+c_4\ln\left(\frac{\tau}{2r_0}\right)\right] \\
&+\int_0^{+\infty}\mathrm{d}\tau\,(\mathrm{M}^{(2)}_{a\langle i}\mathrm{M}^{(5)}_{j\rangle a})(u-\tau)\left[c_5\ln\left(\frac{\tau}{2b_0}\right)+c_6\ln\left(\frac{\tau}{2r_0}\right)\right] \\
&+\int_0^{+\infty}\mathrm{d}\tau\,(\mathrm{M}^{(1)}_{a\langle i}\mathrm{M}^{(6)}_{j\rangle a})(u-\tau)\left[c_7\ln\left(\frac{\tau}{2b_0}\right)+c_8\ln\left(\frac{\tau}{2r_0}\right)\right] \\
&+\int_0^{+\infty}\mathrm{d}\tau\,(\mathrm{M}_{a\langle i}\mathrm{M}^{(7)}_{j\rangle a})(u-\tau)\left[c_9\ln\left(\frac{\tau}{2b_0}\right)+c_{10}\ln\left(\frac{\tau}{2r_0}\right)\right] \\
&+\mathrm{M}^{(2)}_{a\langle i}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{j\rangle a}(u-\tau)\left[c_{11}\ln\left(\frac{\tau}{2r_0}\right)+c_{12}\right] \\
&+\mathrm{M}^{(1)}_{a\langle i}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(6)}_{j\rangle a}(u-\tau)\left[c_{13}\ln\left(\frac{\tau}{2r_0}\right)+c_{14}\right] \\
&+\mathrm{M}_{a\langle i}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(7)}_{j\rangle a}(u-\tau)\left[c_{15}\ln\left(\frac{\tau}{2r_0}\right)+c_{16}\right]\Bigg\}.
\end{aligned}
$$